Regular Hereditary Subcategories

by

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REGULAR HEREDITARY SUBCATEGORIES

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The purpose of this paper is to present several new categorical notions which play a fundamental role in the study of categorical reflections and epireflections. The notions are used to establish sufficient conditions to guarantee that a subcategory will be reflective if and only if it is epireflective.

We call a subcategory $\overline{B}$ of a category $\overline{C}$ regular hereditary in $\overline{C}$ provided that the domain of a regular monomorphism in $\overline{C}$ is an object in $\overline{B}$ whenever the codomain is an object in $\overline{B}$. In the case of complete categories we show that a subcategory $\overline{B}$ of a category $\overline{C}$ is strongly closed under limits in $\overline{C}$ if and only if it is closed under limits and is regular hereditary in $\overline{C}$. Using this we are able to show that in certain categories a regular hereditary subcategory of $\overline{C}$ is reflective in $\overline{C}$ if and only if it is epireflective in $\overline{C}$.

If M is a class of monomorphisms in a category $\overline{C}$, we say that $\overline{C}$ is M-subspanned by a subcategory $\overline{B}$ provided that each object in $\overline{C}$ is the domain of some morphism in M. with codomain an object in $\overline{B}$. A subcategory $\overline{B}$ of a complete category $\overline{C}$ is shown to be regular hereditary in $\overline{C}$ whenever $\overline{C}$ is M-subspanned by $\overline{B}$ and $\overline{B}$ is closed under intersections in $\overline{C}$. As a consequence we show that whenever $\overline{C}$ is a complete category which is M-subspanned by a subcategory $\overline{B}$, that $\overline{B}$ is reflective in $\overline{C}$ if and only if $\overline{B}$ is epireflective in $\overline{C}$. We also show that in special categorical situations there are no non-trivial reflective subcategories of a category $\overline{C}$ containing a subcategory which regularly subspans $\overline{C}$.

Regular hereditariness is used to develop the notion of fitting subcategories with which we are able to characterize the objects in the epireflective hull of certain fitting subcategories. As a corollary to this result we obtain a recent result by S. Franklin.

INTRODUCTION

The purpose of this paper is to present several new categorical notions which we show play a fundamental role in the study of categorical reflections and epireflections. Relatively little is known about reflective subcategories in general. The most outstanding result, due to Freyd, is that reflective subcategories are closed under limits (1.18). H. Herrlich has shown that there are pathological reflective subcategories which are not epireflective; in particular, that the category of all powers of some strongly rigid Hausdorff space is a reflective subcategory of the category of Hausdorff spaces which is not epireflective. The question arises as to when reflectivity for subcategories is equivalent to epireflectivity. In this paper we establish sufficient conditions to guarantee that a subcategory will be reflective if and only if it is epireflective (2.10).

In Chapter I we present the fundamental categorical results essential to the study of reflective and epireflective subcategories. The result showing that epireflective subcategories are closed under extremal subobjects (1.19) partly motivates the definition of extremal hereditary subcategories (2.1) and indicates the role that hereditary conditions will play in the study of reflective and epireflective subcategories.

In Chapter II we define the notions of regular hereditariness and subspanning for subcategories. Using these notions we obtain several previously unknown results (2.3, 2.9) showing conditions for which closure under limits is equivalent to strong closure under limits. With these results we are able to generalize some known results by H. Herrlich and G. Strecker (2.4) and J. van der Slot (2.6). We also are able in special cases to characterize reflective subcategories as those subcategories which are closed under limits (2.10).

In Chapter II we also use the notion of subspanning to prove several results showing that in special categorical situations there are no non-trivial reflective subcategories of a category $\overline{C}$ containing a subcategory which regularly subspans $\overline{C}$ (2.12, 2.13).

In Chapter III we use the notion of regular hereditariness to define fitting subcategories (3.3). The first major result (3.4) relates fitting to closure under limits and strong closure under limits. Consequently, using the results of Chapter II, we are able to show how fitting conditions lead to reflections and epireflections (3.8). We are also able to use fitting conditions to generalize a topological result by J. van der Slot (3.7).

Finally we prove a categorical result (3.9), which characterizes the objects in the epireflective hulls of certain fitting subcategories. Let $\overline{C}$ be the category of completely regular Hausdorff spaces and $\overline{A}$ the category of compact Hausdorff spaces. Let $\overline{B}$ be a subcategory of $\overline{C}$

containing all completely regular Hausdorff spaces satisfying a property P which is inversely preserved under perfect maps. We prove then that $\overline{B}$ is $\overline{A}$-fitting; as a corollary we are able to obtain a recent result by S. Franklin (3.14).

CHAPTER I

Preliminaries

The purpose of this chapter is to present the major categorical notions that will be used throughout this paper. The results that are stated in Chapter I are taken primarily from the work of H. Herrlich and G. Strecker [3, 4, 5]. We provide proofs of some of the results for the sake of completeness.

Definition 1.1: Given a category $\overline{C}$, a source in $\overline{C}$ is a family $(S,s_i)_I$ where S is an object in $\overline{C}$ and for each $i\epsilon I$ s_i is a morphism with domain S. A monosource is a source $(S,s_i)_I$ with the property that whenever p and q are morphisms such that $s_i p=s_i q$ for each $i\epsilon I$, then $p=q$; that is, the family $(s_i)_I$ is "simultaneously left-cancellable."

Definition 1.2: Let $\overline{C}$ be a category and I a small category.

(i) A natural source for a functor D from I to $\overline{C}$ is a source $(L,q_i)_I$ such that $q_i:L\longrightarrow D(i)$ for each $i\epsilon I$ and for each morphism $m:i\longrightarrow j$ the diagram

$$\begin{tikzcd}
 & D(i) \arrow[d, "D(m)"] \\
L \arrow[ur, "q_i"] \arrow[r, "q_j"'] & D(j)
\end{tikzcd}$$

commutes.

(ii) A limit for a functor D is a natural source $(L,q_i)_I$ for D such that for any natural source $(M,m_i)_I$ for D there exists a unique morphism $f:M \longrightarrow L$ such that $q_i f=m_i$ for each $i \in I$.

Definition 1.3: A category $\overline{C}$ is said to be complete provided that for each small category I, every functor from I into $\overline{C}$ has a limit.

Definition 1.4: Let $\overline{C}$ be a category and $f:A \longrightarrow B$ a morphism in $\overline{C}$; then (A,f) is said to be a subobject of B provided that f is a monomorphism in $\overline{C}$.

Definition 1.5: If (A,f) and (A′,g) are subobjects of B, then (A,f) is said to be smaller than (A′,g) (denoted by $(A,f) \leq (A',g)$) if and only if there is some morphism h such that the following diagram

$$\begin{tikzcd}
A \arrow[dr, "f"] \arrow[dd, "h"'] & \\
& B \\
A' \arrow[ur, "g"'] &
\end{tikzcd}$$

commutes. If $(A,f) \leq (A',g)$ and $(A',g) \leq (A,f)$, then (A,f) and (A′,g) are said to be isomorphic subobjects of B; we denote this by $(A,f) \approx (A',g)$.

It is easily shown that $\approx$ is an equivalence relation on the class of all subobjects of an object B. Therefore, the class of all subobjects of an object B may be partitioned into equivalence classes of isomorphic subobjects. Via the Axiom of Choice, choose a system of representatives for the equivalence relation $\approx$ on the class of all subobjects of B. Such a system of representatives will be called a representative class of subobjects of B.

Definition 1.6: A category $\overline{C}$ is said to be well-powered provided that each $\overline{C}$-object has a representative class of subobjects which is a set. (The dual notion is co-well-powered.)

Many times we inaccurately identify the subobject of an object B with its domain. That is, if $f:A \longrightarrow B$ is a monomorphism in $\overline{C}$, we sometimes say that A is a subobject of B.

Definition 1.7: Let $\overline{C}$ be a category.

- (i) A morphism $f:A \longrightarrow B$ is an extremal monomorphism provided that the following conditions hold:
  - (a) f is a monomorphism, and
  - (b) whenever $f=ne$ with e an epimorphism, then e is an isomorphism.
- (ii) If $f:A \longrightarrow B$ is an extremal monomorphism we say that (A,f) (and sometimes just A) is an extremal subobject of B.

Proposition 1.8: In any category $\overline{C}$, if (X,e) is an equalizer for morphisms $f,g:Y \longrightarrow A$, then e is an extremal monomorphism.

Proof: Let $h,k:P \longrightarrow X$ be morphisms such that $eh=ek$. Since $f(eh)=(fe)h=(ge)h=g(eh)$, there exists a unique morphism $q:P \longrightarrow X$ such that $eq=eh$. Since q is a unique, $q=h=k$ and e is an monomorphism.

Let $s:X \longrightarrow Q$ and $r:Q \longrightarrow Y$ be morphisms where $e=rs$ and s is an epimorphism. Then $(fr)s=f(rs)=fe=ge=g(rs)=(gr)s$ and since s is an epimorphism we have $fr=gr$. Thus there exists a morphism $t:Q \longrightarrow X$ such that $et=r$. Now $e(ts)=(et)s=rs=e=e1_X$, and we have $ts=1_X$; so s is a section and an epimorphism. Thus s is an isomorphism.

Remark: The reader should observe that every section $f:X \longrightarrow Y$ equalizes the morphisms fg and 1_Y where g is a morphism such that $gf=1_X$. If (X,e) is an equalizer in a category $\overline{C}$, then e will be called a regular monomorphism in $\overline{C}$.

Definition 1.9: Let $\overline{C}$ be a category and $\{m_i | m_i:X_i \longrightarrow Y\}_I$ be a family of monomorphisms in $\overline{C}$; then $(Q,k_i,d)_I$ will be called the intersection of this family provided that:

(i) $k_i:Q \longrightarrow X_i$ for each $i \in I$,

(ii) $d:Q \longrightarrow Y$,

(iii) $d=m_ik_i$ for each $i \in I$,

and whenever (T,p_i,f) satisfies analagous conditions, then there exists a unique morphism $g:T \longrightarrow Q$ such that $k_ig=p_i$ for each $i \in I$.

Definition 1.10:

(i) A category is said to have the epi-extremal mono factorization property provided that each morphism $f:X \longrightarrow Y$ can be written as the composition $f=me$ of two morphisms e and m where e is an epimorphism and m is an extremal monomorphism.

(ii) A category has the unique epi-extremal mono factorization property provided that it has the epi-extremal mono factorization property and whenever $f=me=m'e'$ with m and m' extremal monomorphisms and e and e' epimorphisms, then there exists an isomorphism k such that the following diagram

$$\begin{tikzcd} {} \arrow[r, "e"] \arrow[d, "e'"'] & {} \arrow[d, "m"] \arrow[dl, "k"'] \\ {} \arrow[r, "m'"'] & {} \end{tikzcd}$$

commutes.

(iii) Given a monomorphism $m:X \longrightarrow Y$, then a subobject $g:Z \longrightarrow Y$ is said to contain X provided that $(X,m) \leq (Z,g)$.

Lemma 1.11: Let $\overline{C}$ be a complete and well-powered category, $f:X \longrightarrow Y$ a morphism, and M a class of monomorphisms in $\overline{C}$ with the following properties:

(x) M is closed under intersections,

(y) whenever $f=mkg$ where $m \in M$ and k is a regular monomorphism, then $mk \in M$ and

(z) $1_Y \in M$.

Then there exist morphisms e and $m \in M$ with e an epimorphism such that:

(i) $f=me$;

(ii) if $f=m'e'$ with $m'\epsilon M$ there exists a morphism k such that the diagram

$$\begin{tikzcd}
{} \arrow[r, "e"] \arrow[d, "e'"'] & {} \arrow[d, "m"] \arrow[dl, "k"'] \\
{} \arrow[r, "m'"'] & {}
\end{tikzcd}$$

commutes;

(iii) $e=\overline{m}\overline{e}$ with $m\overline{m}\epsilon M$ implies that $\overline{m}$ is an isomorphism.

Proof: Since $\overline{C}$ is well-powered and $f=1_Y f$ with $1_Y\epsilon M$ there exists an nonempty representative set $\{m_i e_i=f|i\epsilon I\}$ of factorizations of f where $m_i\epsilon M$ for each $i\epsilon I$. Let $(D,q_i,m)_I$ be the intersection of the set of monomorphisms $m_i:X_i\longrightarrow Y$; then by (x), m is a morphism in M. Moreover, there exists a morphism $e:X\longrightarrow D$ such that $f=me$ and condition (i) is satisfied. Suppose $f=m'e'$ where $m'\epsilon M$. Then up to an isomorphism $m'=m_i$ and $e'=e_i$ for some $i\epsilon I$, and there exists a morphism k such that $m=m_i k$. Also $m_i ke=me=m'e'=m_i e'$. Thus $ke=e'$, since $m_i\epsilon M$ is a monomorphism, so that the following diagram

$$\begin{tikzcd}
{} \arrow[r, "e"] \arrow[d, "e_i=e'"'] & {} \arrow[d, "m"] \arrow[dl, "k"'] \\
{} \arrow[r, "m'=m_i"'] & {}
\end{tikzcd}$$

commutes. Hence condition (ii) is satisfied.

Suppose $e=\overline{m}\overline{e}$ and $m\overline{m}\epsilon M$. Then there exists a morphism t such that the following diagram

$$\begin{tikzcd}
{} \arrow[rr, "e"] \arrow[d, "\overline{e}"'] & & {} \arrow[d, "m"] \arrow[dll, "t"'] \\
{} \arrow[r, "\overline{m}"'] & {} \arrow[r, "m"'] & {}
\end{tikzcd}$$

commutes. Thus $m\overline{m}t=m=m1$ and since m is a monomorphism we have $\overline{m}t=1$. Now $\overline{m}$ is a retraction and since $m\overline{m}\in M$, $\overline{m}$ is also a monomorphism, therefore $\overline{m}$ is a isomorphism. Condition (iii) follows.

All that remains to show is the requirement that e be an epimorphism. Let r and s be morphisms such that $re=se$. Let (E,v) be the equalizer of the morphisms r and s. Then there exists a unique morphism k such that $vk=e$; thus by (y), $mv\in M$. From (iii) we have v is an isomorphism; thus $r=s$ and e is an epimorphism.

Lemma 1.12: Every complete, well-powered category $\overline{C}$ has the property that whenever the diagram

$$\begin{tikzcd}
\cdot \arrow[r, "e"] \arrow[d, "h"'] & \cdot \arrow[d, "g"] \\
\cdot \arrow[r, "m"'] & \cdot
\end{tikzcd}$$

commutes, with e an epimorphism and m an extremal monomorphism, then there exists a morphism k such that $mk=g$ and $ke=h$.

Proof: Let

$$\begin{tikzcd}
\cdot \arrow[r, "e"] \arrow[d, "h"'] & \cdot \arrow[d, "g"] \\
\cdot \arrow[r, "f"'] & \cdot
\end{tikzcd}$$

be a commutative diagram with e an epimorphism and f an extremal monomorphism. Let M be the class of all monomorphisms m so that there exist morphisms f_m and g_m such that the following diagram

$$\begin{tikzcd}
{} \arrow[rr, "e"] \arrow[dd, "h"'] & & {} \arrow[dd, "g"] \arrow[dl, "g_m"'] \\
& {} \arrow[dr, "m"] & \\
{} \arrow[ur, "f_m"] \arrow[rr, "f"'] & & {}
\end{tikzcd}$$

commutes. M can easily be shown to satisfy the conditions of Lemma 1.11. Thus there exists a factorization $f=m\bar{e}$ of f where $m\epsilon M$ and $\bar{e}$ is an epimorphism. The morphism f is an extremal monomorphism so it follows that $\bar{e}$ is an isomorphism. By the construction of M, there is a morphism g_m so that

$$\begin{tikzcd}
{} \arrow[rr, "e"] \arrow[dd, "h"'] & & {} \arrow[dd, "g"] \arrow[dl, "g_m"'] \\
& {} \arrow[dr, "m"] & \\
{} \arrow[ur, "\bar{e}"] \arrow[rr, "f"'] & & {}
\end{tikzcd}$$

commutes. Let $k=\bar{e}^{-1}g_m$ and the proof is complete.

Lemma 1.13: Let $\bar{C}$ be a complete, well-powered category; then the class of all extremal monomorphisms is closed under composition and intersections.

Proof: Let $m_1:X\longrightarrow Y$ and $m_2:Y\longrightarrow Z$ be extremal monomorphisms. Let $f=m_2m_1$ and suppose that $f=me$ where e is an epimorphism. Then there exists by Lemma 1.12 a morphism k such that the following diagram

$$\begin{tikzcd}
{} \arrow[r, "e"] \arrow[d, "m_1"'] & {} \arrow[d, "m"] \arrow[dl, "k"] \\
{} \arrow[r, "m_2"'] & {}
\end{tikzcd}$$

commutes. The morphism $m_1=ke$ is an extremal monomorphism; consequently e must be an isomorphism.

Let $m_i:X_i\longrightarrow Y$, $i\in I$, be a class of extremal monomorphisms and $(L,s_i,d)_I$ their intersection. We will show that d is an extremal monomorphism. Suppose $d=me$ where e is an epimorphism; then by Lemma 1.12, for each $i\in I$ there exists a morphism q_i such that the following diagram

$$\begin{tikzcd}
 & Z \arrow[rd, "m"] \arrow[dd, "q_i"] & \\
L \arrow[ru, "e"] \arrow[rd, "s_i"'] & & Y \\
 & X_i \arrow[ru, "m_i"'] &
\end{tikzcd}$$

commutes. Consequently, there exists a morphism $q:Z\longrightarrow L$ such that $s_iq=q_i$ for each $i\in I$. Then $s_iqe=q_ie=s_i=s_i1_L$; thus we have $qe=1_L$, since $(L,s_i)_I$ is a monosource. Hence e is a section and an epimorphism and thus an isomorphism.

Lemma 1.14: If $\overline{C}$ is a complete, well-powered category, then $\overline{C}$ has the unique epi-extremal mono factorization property.

Proof: The class M of all extremal monomorphisms satisfies conditions (x), (y) and (z) in Lemma 1.11. Thus $\overline{C}$ has the epi-extremal mono factorization property.

To show that $\overline{C}$ has the unique epi-extremal mono factorization property, suppose that $f=m_1e_1=m_2e_2$ where e_1 and e_2 are epimorphisms and m_1 and m_2 are extremal monomorphisms. By Lemma 1.12, there exists a morphism h such that the following diagram

$$\begin{tikzcd} {} \arrow[r, "e_1"] \arrow[d, "e_2"'] & {} \arrow[d, "m_1"] \arrow[dl, "h"'] \\ {} \arrow[r, "m_2"'] & {} \end{tikzcd}$$

commutes and a morphism k such that the diagram

$$\begin{tikzcd} {} \arrow[r, "e_1"] \arrow[d, "e_2"'] & {} \arrow[d, "m_1"] \\ {} \arrow[r, "m_2"'] \arrow[ur, "k"] & {} \end{tikzcd}$$

commutes.

Now $m_1(kh)=(m_1k)h=m_2h=m_1=m_1 1$ and since m_1 is a monomorphism, we have kh=1. Also $m_2(hk)=(m_2h)k=m_1k=m_2=m_2 1$ and since m_2 is a monomorphism, hk=1. Thus h is an isomorphism and the lemma follows.

Throughout the remainder of this paper all subcategories will be assumed to be full and isomorphism-closed unless otherwise stated. That is, if $\overline{A}$ is a subcategory of $\overline{B}$, X and Y objects in $\overline{A}$ and $f:X\longrightarrow Y$ a morphism in $\overline{B}$, then f is in $\overline{A}$, and if $g:A\longrightarrow Y$ is an isomorphism with A an object in $\overline{A}$, then Y is an object in $\overline{A}$.

Definition 1.15: If $\overline{A}$ is a subcategory of the category $\overline{C}$ and if for each object X in $\overline{C}$ there exists an object $X_{\overline{A}}$ in $\overline{A}$ and a morphism $r:X\longrightarrow X_{\overline{A}}$ such that for each morphism $f:X\longrightarrow A$ where A is an object in $\overline{A}$, there exists a unique morphism $f':X_{\overline{A}} \longrightarrow A$ such that the following diagram

$$\begin{tikzcd}
X \arrow[r, "r"] \arrow[dr, "f"'] & X_{\overline{A}} \arrow[d, "f'"] \\
& A
\end{tikzcd}$$

commutes, then $\overline{A}$ is said to be a reflective subcategory of $\overline{C}$. The morphism r is called a reflection morphism from X to $X_{\overline{A}}$; $X_{\overline{A}}$ will be called the reflection of X in $\overline{A}$. If M is a class of morphisms and each reflection morphism is in M, $\overline{A}$ is said to be an M-reflective subcategory of $\overline{C}$. In particular, if each reflection is a monomorphism (epimorphism) $\overline{A}$ is said to be monoreflective (epireflective) in $\overline{C}$.

Proposition 1.16: Every monoreflective subcategory $\overline{A}$ of a category $\overline{C}$ is epireflective.

Proof: Let $r:X \longrightarrow X_{\overline{A}}$ be any reflection morphism. Let $f,g:X_{\overline{A}} \longrightarrow Y$ be morphisms in $\overline{C}$ such that $fr=gr$. Y is an object in $\overline{C}$; therefore, there exists a reflection morphism $r':Y \longrightarrow Y_{\overline{A}}$. Now $(r'f)r=r'(fr)=r'(gr)=(r'g)r$. Thus $r'f$ and $r'g$ are morphisms such that the following diagram

$$\begin{tikzcd}
X \arrow[r, "r"] \arrow[dr, "(r'f)r=(r'g)r"'] & X_{\overline{A}} \arrow[d, "r'f \text{ or } r'g"] \\
& Y_{\overline{A}}
\end{tikzcd}$$

commutes. From the uniqueness condition in Definition 1.15 we conclude that $r'f=r'g$. We now observe that r' is a monomorphism, thus $f=g$. Therefore, r is an epimorphism and the proof is complete.

Proposition 1.17: Every reflective subcategory of any category is closed under the formation of retracts.

Proof: Suppose $\overline{A}$ is a reflective subcategory of the category $\overline{C}$ and X a retract of an object A in $\overline{A}$. Let $f:X\longrightarrow A$ and $g:A\longrightarrow X$ be morphisms such that $gf=1_X$. Let $r:X\longrightarrow X_{\overline{A}}$ be the reflection morphism for X. By Definition 1.15 there exists a morphism $g':X_{\overline{A}}\longrightarrow A$ such that the following diagram

$$\begin{tikzcd}
X \arrow[r, "r"] \arrow[dr, "f"'] & X_{\overline{A}} \arrow[d, "g'"] \\
 & A
\end{tikzcd}$$

commutes. Now $(gg')r=g(g'r)=gf=1_X$; moreover, the morphisms $1_{X_{\overline{A}}}$ and rgg' make the following diagram

$$\begin{tikzcd}
X \arrow[r, "r"] \arrow[dr, "r"'] & X_{\overline{A}} \arrow[d, "rgg' \text{ or } 1_{X_{\overline{A}}}"] \\
 & X_{\overline{A}}
\end{tikzcd}$$

commute. Hence by the uniqueness condition in Definition 1.15, $r(gg')=1_{X_{\overline{A}}}$. Consequently X is isomorphic to $X_{\overline{A}}$ and X is an object in $\overline{A}$.

Theorem 1.18: [Freyd: 2, p. 79] Let I be a small category and $\overline{A}$ be a reflective subcategory of $\overline{C}$. If a functor $D:I\longrightarrow\overline{A}$ has a limit $(L,q_i)_I$ in $\overline{C}$, then L is in $\overline{A}$.

Proof: Let $r:L\longrightarrow L_{\overline{A}}$ be the reflection morphism for L. Then for each $i\in I$ there exists a morphism $q_i':L_{\overline{A}}\longrightarrow D(i)$

such that $q_i{}'r=q_i$, and for each morphism $m:i\longrightarrow j$ in I the diagram

$$\begin{tikzcd}
 & & D(i) \arrow[dd, "D(m)"] \\
L \arrow[urr, "q_i"] \arrow[r, "r"] \arrow[drr, "q_j"'] & L_{\overline{A}} \arrow[ur, "q_i{}'"] \arrow[dr, "q_j{}'"'] & \\
 & & D(j)
\end{tikzcd}$$

commutes, since $D(m)q_i{}'r=D(m)q_i=q_j=q_j{}'r$ and from Definition 1.15, $D(m)q_i{}'=q_j$. Hence $(L_{\overline{A}},q_i{}')_I$ is a natural source for the functor $D:I\longrightarrow\overline{A}$, so there exists a morphism $f:L_{\overline{A}}\longrightarrow L$ such that for each $i\epsilon I$, $q_if=q_i{}'$. Therefore, $q_i(fr)=(q_if)r=q_i{}'r=q_i=q_i1_L$ and since $(L,q_i)_I$ is a monosource we have $fr=1_L$. Moreover, $1_{L_{\overline{A}}}$ and rf make the following diagram

$$\begin{tikzcd}
L \arrow[r, "r"] \arrow[dr, "r"'] & L_{\overline{A}} \arrow[d, "1_{L_{\overline{A}}} \text{ or } rf"] \\
 & L_{\overline{A}}
\end{tikzcd}$$

commute. Thus $rf=1_{L_{\overline{A}}}$; hence r is an isomorphism, so that L is an object in $\overline{A}$.

Proposition 1.19: Every epireflective subcategory of any category is closed under the formation of extremal subobjects.

Proof: Let $\overline{A}$ be an epireflective subcategory of the category $\overline{C}$ and $f:X\longrightarrow A$ an extremal monomorphism with A an object in $\overline{A}$. Let $r:X\longrightarrow X_{\overline{A}}$ be the reflection morphism for X in $\overline{A}$; then there exists a morphism $g:X_{\overline{A}}\longrightarrow A$ such that the following diagram

$$\begin{tikzcd} X \arrow[r, "r"] \arrow[dr, "f"'] & X_{\overline{A}} \arrow[d, "g"] \\ & \overline{A} \end{tikzcd}$$

commutes. Since r is an epimorphism and f is an extremal monomorphism, r is an isomorphism and X is an object in $\overline{A}$.

Theorem 1.20: If $\overline{C}$ is a category which is

(i) co-well-powered,

(ii) has products and

(iii) has the epi-extremal mono factorization property,

and if $\overline{A}$ is a subcategory of $\overline{C}$, then the following are equivalent:

(i′) $\overline{A}$ is epireflective in $\overline{C}$,

(ii′) $\overline{A}$ is closed under the formation of products and extremal subobjects.

Proof: Theorem 1.18 and Proposition 1.19 show that (i′) implies (ii′). Let X be an object in $\overline{C}$ and $\{g_i : X \longrightarrow A_i\}_I$ be a representative set of quotient objects of X which belong to $\overline{A}$. Let $(P, p_i)_I$ be the product of the objects $(A_i)_I$. The object P is in the category $\overline{A}$ by (ii′), and there exists a morphism $g : X \longrightarrow P$ such that the following diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[dr, "g_i"'] & P \arrow[d, "p_i"] \\ & A_i \end{tikzcd}$$

commutes for each $i \in I$. Let

$$\begin{tikzcd}
X \arrow[rr, "g"] \arrow[dr, "r"'] & & P \\
& X_{\overline{A}} \arrow[ur, "h"'] &
\end{tikzcd}$$

be an epi-extremal mono factorization of g. By (ii′) the object $X_{\overline{A}}$ is in $\overline{A}$. We show that $r:X \longrightarrow X_{\overline{A}}$ is the reflection for X in $\overline{A}$.

Let A' be any object in $\overline{A}$ and $f:X \longrightarrow A'$ a morphism in $\overline{C}$. Let

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "t"'] & & A' \\
& D \arrow[ur, "p"'] &
\end{tikzcd}$$

be an epi-extremal mono factorization of the morphism f. D is an object in $\overline{A}$ by (ii′), so since $\{g_i:X \longrightarrow A_i\}_I$ is a representative class of all quotient objects of X, we can assume that $t:X \longrightarrow D$ is the same as $g_j:X \longrightarrow A_j$ for some $j \in I$. Thus the following diagram

$$\begin{tikzcd}
& X \arrow[r, "r"] \arrow[dl, "f"'] \arrow[d, "g_j"] \arrow[dr, "g"] & X_{\overline{A}} \arrow[d, "h"] \\
A' & A_j \arrow[l, "p"] & P \arrow[l, "p_j"]
\end{tikzcd}$$

commutes. Let $f'=pp_jh$; then $f':X_{\overline{A}} \longrightarrow A'$ and $f'r=(pp_jh)r=pp_j(hr)=pp_jg=pg_j=f$. The uniqueness of f' follows from the fact that r is an epimorphism.

Corollary 1.21: If $\overline{C}$ is a category which is co-well-powered, has products and the epi-extremal mono factorization property, then the intersection of any collection of epireflective subcategories of $\overline{C}$ is an epireflective subcategory of $\overline{C}$.

Proposition 1.22: If $\overline{C}$ is a category which

(i) is co-well-powered,

(ii) has products and

(iii) has the epi-extremal mono factorization property,

and if $\overline{A}$ is any subcategory of $\overline{C}$, then there exists a smallest epireflective subcategory $\overline{A}'$ of $\overline{C}$ containing $\overline{A}$, called the epireflective hull of $\overline{A}$ in $\overline{C}$. Furthermore, if $\overline{C}$ is complete and well-powered then the objects of $\overline{A}'$ are precisely the extremal subobjects of products of objects in $\overline{A}$.

Proof: $\overline{C}$ is clearly an epireflective subcategory of $\overline{C}$ containing $\overline{A}$. Corollary 1.21 guarantees the existence of a smallest epireflective subcategory $\overline{A}'$ of $\overline{C}$ which contains $\overline{A}$, namely the intersection of all the epireflective subcategories of $\overline{C}$ which contain $\overline{A}$.

Now suppose that $\overline{C}$ is complete and well-powered; then let $\overline{B}$ be the subcategory of $\overline{C}$ whose objects are all extremal subobjects of products of objects in $\overline{A}$. We show that $\overline{B}=\overline{A}'$. It is clear that $\overline{A}$ is a subcategory of $\overline{B}$ since the identity morphism is an extremal monomorphism and $(A,1_A)$ is the product of the family containing the single object A. Theorem 1.20 shows that $\overline{A}'$ is closed under products and extremal subobjects; thus it is clear that $\overline{B}$ is contained in $\overline{A}'$.

Therefore it suffices to show that $\overline{B}$ is closed under products and extremal subobjects to conclude that $\overline{B}$ is an epireflective subcategory of $\overline{C}$ containing $\overline{A}$ and thus that $\overline{A}'$ is contained in $\overline{B}$.

Let $(X,v_i)_I$ be a product where the codomain of each v_i is an object X_i in the category $\overline{B}$. For each $i\epsilon I$ there exists an object S_i and an extremal monomorphism $q_i:X_i \longrightarrow S_i$ where S_i is a product of objects in $\overline{A}$. For each i in I let $(S_i,u_j^i)_{J(i)}$ be a product in $\overline{C}$ where $u_j^i:S_i \longrightarrow S_j^i$, and each S_j^i is an object in $\overline{A}$. Let $K=\{(i,j)\,|\,i\epsilon I \text{ and } j\epsilon J(i)\}$. Let $(S,w_j^i)_K$ be the product of all the S_j^i. By the definition of product there exists for each $i\epsilon I$ a morphism $f_i:S \longrightarrow S_i$ such that the following diagram

$$\begin{tikzcd}
S \arrow[r, "f_i"] \arrow[dr, "w_j^i"'] & S_i \arrow[d, "u_j^i"] \\
 & S_j^i
\end{tikzcd}$$

commutes for each $j\epsilon J(i)$.

That $(S,f_i)_I$ is a product is clear; consequently there exists a morphism $e:X \longrightarrow S$ such that the diagram

$$\begin{tikzcd}
X \arrow[r, "e"] \arrow[d, "v_i"'] & S \arrow[d, "f_i"] \\
X_i \arrow[r, "q_i"'] & S_i
\end{tikzcd}$$

commutes.

To conclude that X is an object in $\overline{B}$, it suffices to prove that e is an extremal monomorphism. Suppose $e=mr$ where $r:X\longrightarrow T$ is an epimorphism. By Lemma 1.12, there exists a morphism $t_i:T\longrightarrow X_i$ for each $i\in I$, such that the following diagram

$$\begin{tikzcd}
X \arrow[rr, "e"] \arrow[dr, "r"] \arrow[dd, "v_i"'] & & S \arrow[dd, "f_i"] \\
& T \arrow[ur, "m"'] \arrow[dl, "t_i"'] & \\
X_i \arrow[rr, "q_i"'] & & S_i
\end{tikzcd}$$

commutes.

From the definition of product, there exists a morphism $u:T\longrightarrow X$ such that $v_i u=t_i$ for each $i\in I$. Now $q_i(v_i ur)=q_i(v_i u)r=q_i t_i r=q_i v_i=q_i v_i 1_X$ for each $i\in I$; and since each q_i is a monomorphism, we have $v_i ur=v_i 1_X$. Moreover (X,v_i) is a monosource since it is a product; thus $ur=1_X$. Consequently, r is an epimorphism and a section, and thus an isomorphism.

Lemma 1.13 shows that the composition of extremal monomorphisms in $\overline{C}$ is an extremal monomorphism; thus $\overline{B}$ is closed under extremal subobjects. Consequently $\overline{B}$ is an epireflective subcategory of $\overline{C}$ containing $\overline{A}$ and we have $\overline{A}'\subseteq\overline{B}$.

Much of the work in this paper deals with epireflective subcategories and the notion of strong closure under limits. We now state as our final definition in Chapter I: the notion of strong closure under limits.

Definition 1.23: Let I be a small category, $\overline{C}$ be any category and $\overline{A}$ a subcategory of $\overline{C}$. A functor $D:I\longrightarrow\overline{C}$ is said to be partially in $\overline{A}$ provided that, for each object i in I, there exists some $j \in I$ such that D(j) is in $\overline{A}$ and there is a morphism $m: j \longrightarrow i$.

Definition 1.24: A subcategory $\overline{A}$ of a category $\overline{C}$ is said to be strongly closed under limits in $\overline{C}$ provided that each limit in $\overline{C}$ of every small functor which is partially in $\overline{A}$ is itself in $\overline{A}$.

We complete Chapter I by examining the categories of topological spaces and Hausdorff spaces. The proofs of Proposition 1.25 and Theorem 1.27 are well known and will be omitted.

Proposition 1.25: In the category of topological spaces (respectively Hausdorff spaces):

(i) isomorphism means homeomorphism,
(ii) monomorphism means one to one continuous function,
(iii) epimorphism means continuous onto (respectively dense) function,
(iv) retraction means topological retraction map,
(v) product means topological product,
(vi) coproduct means disjoint topological union.

The fundamental importance of the notion of extremal monomorphism in the study of reflections can be seen above. We now characterize the extremal monomorphisms in the category of topological spaces and in the category of Hausdorff spaces. We include the proof of this theorem because of the importance of the construction of equalizers in these categories.

Theorem 1.26: In the category of topological spaces (respectively Hausdorff spaces) the following are equivalent:

(i) equalizer,

(ii) extremal monomorphism,

(iii) topological embedding (respectively closed embedding).

Proof: Proposition 1.8 shows that (i) implies (ii). So suppose that $f:X\longrightarrow Y$ is an extremal monomorphism. Let $Z=f[X]$ (respectively $\overline{f[X]}$) with the relative topology. Let $f':X\longrightarrow Z$ be defined by $f'(x)=f(x)$ for all $x\epsilon X$ and $i:Z\longrightarrow Y$ be the inclusion function. f' is onto (respectively dense) so f' is an epimorphism. Since f is an extremal monomorphism f' must be an isomorphism, consequently f' is a homeomorphism and thus an embedding (respectively a closed embedding).

Now suppose that $f:X\longrightarrow Y$ is a topological embedding (respectively closed embedding). Let $Y_1=Y_2=Y$ and $Z=Y_1\oplus Y_2$ be the disjoint topological union of Y_1 and Y_2 with injection maps $u_1,u_2:Y\longrightarrow Z$. Define an equivalence relation R on Z as follows.

> For each $y\epsilon f[X]$ (respectively $y\epsilon\overline{f[X]}$) let $u_1(y)Ru_2(y)$ and $u_2(y)Ru_1(y)$, and for each $z\epsilon Z$ let zRz.

Let $W=Z/R$ with the quotient topology and $g:Z\longrightarrow W$ be the canonical map. It is clear that $(gu_1)f=(gu_2)f$. Suppose $h:S\longrightarrow Y$ is a map such that $(gu_1)h=(gu_2)h$; then for each $s\epsilon S$, $g(u_1h(s))=g(u_2h(s))$ and $u_1h(s)Ru_2h(s)$ so $h[S]\subset f[X]$ (respectively $h[S]\subset\overline{f[X]}$). Since f is an embedding (closed embedding), let q be the inverse of f; then $t=qh:S\longrightarrow X$ is a continuous map and the unique morphism whose composition with f is h. The proof is complete.

Theorem 1.27: The categories Top (of topological spaces) and Haus (of Hausdorff spaces):

(i) have products,

(ii) have equalizers,

(iii) have coproducts,

(iv) have coequalizers,

(v) are complete,

(vi) are cocomplete,

(vii) are well-powered,

(viii) are co-well-powered,

(ix) are (epi, extremal mono) categories,

(x) are (extremal epi, mono) categories.

We complete this chapter by making a few remarks concerning topological embedding maps in the categories Top and Haus. The remarks are well known and easily verified. If gf is a topological embedding map then f is a topological embedding map. If gf is an epimorphism in either Top or Haus and g is a topological embedding map then f is an epimorphism. The topological embedding maps are preserved under pullbacks and intersections.

CHAPTER II

Hereditary Subcategories

Throughout this chapter we will be concerned with two categorical notions which play a primary role in giving new categorical results. We now state two definitions.

Definition 2.1: Let $\overline{C}$ be a category and M a class of monomorphisms in $\overline{C}$; then if $\overline{B}$ is a subcategory of $\overline{C}$ we say that $\overline{B}$ *is M-hereditary in* $\overline{C}$ if $\overline{B}$ is closed under the formation of M-subobjects in $\overline{C}$. If M is the class of all regular monomorphisms (respectively extremal monomorphisms) in $\overline{C}$ we say that $\overline{B}$ *is regular* (respectively *extremal*) *hereditary in* $\overline{C}$.

Definition 2.2: Let $\overline{C}$ be a category and M a class of monomorphisms in $\overline{C}$; then if $\overline{B}$ is a subcategory of $\overline{C}$ we say $\overline{C}$ is M-*subspanned by* $\overline{B}$ provided each $\overline{C}$-object is an M-subobject in $\overline{C}$ of some object in $\overline{B}$. If M is the class of all monomorphisms (respectively, all regular morphisms, or all extremal monomorphisms), then we say $\overline{C}$ *is subspanned* (respectively *regularly subspanned*, or *extremally subspanned*) *by* $\overline{B}$.

P. Freyd [2] has shown that if $\overline{B}$ is a reflective subcategory of a category $\overline{C}$, then $\overline{B}$ is closed under limits (Theorem 1.18). H. Herrlich and G. Strecker have shown that in the case of epireflective subcategories one has strong closure under limits [4, Theorem 7]. The following question arises. Under what conditions is closure under limits equivalent to strong closure under limits? We now show that regular hereditariness is sufficient to guarantee their equivalence in complete categories. We state this as our first theorem.

Theorem 2.3: If $\overline{C}$ is a complete category and $\overline{B}$ is a subcategory of $\overline{C}$, then the following notions are equivalent:

(i) $\overline{B}$ is closed under limits and regular hereditary in $\overline{C}$,

(ii) $\overline{B}$ is strongly closed under limits.

Proof: Since a subcategory which is strongly closed under limits is clearly regular hereditary, it suffices to prove that (i) implies (ii). Suppose that I is a small category and $D:I\longrightarrow\overline{C}$ is a functor which is partially in $\overline{B}$. Let $J=\{i\in I|D(i)\notin\overline{B}\}$; we assume that J is nonempty, for otherwise the proof is complete. Let $(P,p_i)_{I\setminus J}$ and $(Q,q_i)_J$ be the product in $\overline{C}$ of the objects $\{D(i)\}_{I\setminus J}$ and $\{D(j)\}_J$ respectively. For each $i\in I\setminus J$, $D(i)$ is an object in $\overline{B}$, thus by (i), P is an object in $\overline{B}$. Since the functor $D:I\longrightarrow\overline{C}$ is partially in $\overline{B}$, for each $j\in J$ there exists a morphism $m:i\longrightarrow j$ with domain i in $I\setminus J$. Via the Axiom of Choice, for each $j\in J$ choose one such morphism $m_j:i\longrightarrow j$. We denote the domain of m_j by i_j.

The set of morphisms $f_j=D(m_j)p_{i_j}:P\longrightarrow D(j)$, $j\in J$, induces a unique morphism $t:P\longrightarrow Q$ such that the following diagram

$$\begin{tikzcd}
P \arrow[r, "p_{i_j}"] \arrow[d, "t"'] & D(i_j) \arrow[d, "D(m_j)"] \\
Q \arrow[r, "q_j"'] & D(j)
\end{tikzcd}$$

commutes for each $j\in J$. Let $(L,s_i)_I$ be the limit in $\overline{C}$ of the functor $D:I\longrightarrow\overline{C}$. For each $i\in I\setminus J$ there exists a morphism $s_i:L\longrightarrow D(i)$; thus from the definition of product in $\overline{C}$, there exists a unique morphism $f:L\longrightarrow P$ such that $p_if=s_i$ for each $i\in I\setminus J$. Also, for each $j\in J$ there exists a morphism $s_j:L\longrightarrow D(j)$; hence there exists a morphism $g:L\longrightarrow Q$ such that $q_jg=f_j$ for each $j\in J$. We now show for each $j\in J$, that the following diagram

$$\begin{tikzcd}
P \arrow[rr, "p_{i_j}"] \arrow[dd, "t"'] & & D(i_j) \arrow[dd, "D(m_j)"] \\
& L \arrow[ul, "f"'] \arrow[ur, "s_{i_j}"] \arrow[dl, "g"] \arrow[dr, "s_j"'] & \\
Q \arrow[rr, "q_j"'] & & D(j)
\end{tikzcd}$$

commutes, by showing that $tf=g$.

For each $j\in J$, we have $q_j(tf)=(q_jt)f=(D(m_j)p_{i_j})f=D(m_j)(p_{i_j}f)=D(m_j)s_{i_j}=s_j=q_jg$ by the choice of the $m_j:i_j\longrightarrow j$ and because $(L,s_i)_I$ is a natural source for the functor $D:I\longrightarrow\overline{C}$. Since (Q,q_j) is a monosource, it follows that $tf=g$.

For notational purposes we define $\alpha_i=p_i$ if $i\in I\setminus J$ and $\alpha_i=q_it$ if $i\in J$. Now $(P,\alpha_i)_I$ is a source with domain P and codomain $(D(i))_I$. We wish to show that for any morphism $m:i \longrightarrow k$ the following diagram

$$\begin{tikzcd}
& P \arrow[r, "\alpha_i"] & D(i) \arrow[dd, "D(m)"] \\
L \arrow[ur, "f"] \arrow[urr, "s_i"'] \arrow[drr, "s_k"] \arrow[dr, "f"'] & & \\
& P \arrow[r, "\alpha_k"'] & D(k)
\end{tikzcd}$$

commutes.

Since $D(m)s_i=s_k$ for all $m:i \longrightarrow k$, it suffices to show that $\alpha_h f=s_h$ for all $h \in I$.

Let $h \in I \setminus J$; then $\alpha_h=p_h$ and $\alpha_h f=p_h f=s_h$. Suppose that $h \in J$. Then $\alpha_h=q_h t$ and $\alpha_h f=q_h tf=D(m_h)p_{i_h} f=D(m_h)s_{i_h}=s_h$. Thus the specified diagram commutes since $D(m)\alpha_i f=D(m)s_i=\alpha_k=\alpha_k f$ for each $m:i \longrightarrow k$.

Now for each morphism $m:i \longrightarrow k$ with $i,k \in I$, let (E_m,e_m) be the equalizer of the morphisms $D(m)\alpha_i, \alpha_k : P \longrightarrow D(k)$. By definition each morphism $e_m:E_m \longrightarrow P$ is a regular monomorphism in $\overline{C}$. P is an object in $\overline{B}$ and $\overline{B}$ is regular hereditary in $\overline{C}$; thus we have for each morphism $m:i \longrightarrow k$, the object E_m as an object in the category $\overline{B}$. Let $(T,k_m,d)_M$ be the intersection in $\overline{C}$ of the morphisms $e_m:E_m \longrightarrow P$, $m \in I$. By (i), the object T is in the category $\overline{B}$ and the following diagram

$$\begin{tikzcd}
& E_m \arrow[dr, "e_m"] & \\
T \arrow[ur, "k_m"] \arrow[rr, "d"'] & & P
\end{tikzcd}$$

commutes for any $m \in I$.

Previously we have shown that $D(m)\alpha_i f=\alpha_k f$ for each $m\in I$; thus from the definition of equalizer there exists for each morphism $m\in I$ a unique morphism $h_m: L\longrightarrow E_m$ such that $e_m h_m=f$. Moreover, from the definition of intersection in $\overline{C}$, there exists a unique morphism $s: L\longrightarrow T$ such that the following diagram

$$\begin{tikzcd}
T \arrow[dr, "k_m"] \arrow[drr, "d"] & & \\
& E_m \arrow[r, "e_m"] & P \\
L \arrow[uu, "s"] \arrow[ur, "h_m"'] \arrow[urr, "f"'] & &
\end{tikzcd}$$

commutes for all $m\in I$.

For each $i\in I$, $\alpha_i d: T\longrightarrow D(i)$ is a morphism with codomain $D(i)$, having the property that for each $m: i\longrightarrow k$, $(D(m)\alpha_i)d=(D(m)\alpha_i)e_m k_m=(D(m)\alpha_i e_m)k_m=(\alpha_k e_m)k_m=\alpha_k d$. Thus $(T,\alpha_i d)_I$ is a natural source for the functor $D: I\longrightarrow\overline{C}$. From the definition of limit there exists a unique morphism $r: T\longrightarrow L$ such that the following diagram

$$\begin{tikzcd}
T \arrow[r, "r"] \arrow[d, "\alpha_i d"'] & L \arrow[dl, "s_i"] \\
D(i) &
\end{tikzcd}$$

commutes for each $i\in I$.

We now show that $sr=1_T$ and $rs=1_L$. For each $i\in I\setminus J$ we have $p_i(dsr)=p_i(ds)r=p_i fr=(p_i f)r=s_i r=\alpha_i d=p_i d=p_i(d1_T)$. $(P,p_i)_I$ is a monosource; thus $d(sr)=d1_T$ and since d is a monomorphism we have $sr=1_T$.

To show that $rs=1_L$ we consider two cases; however, we first observe that for each $k \in I$,

$$s_k(rs)=(s_k r)s=(\alpha_k d)s=\alpha_k f.$$

Case 1: Suppose $k \in I \setminus J$. Then $\alpha_k = p_k$ and we have $\alpha_k f = p_k f = s_k$.

Case 2: Suppose $k \in J$. Then $\alpha_k = q_k t$ and we have $\alpha_k f = q_k t f = q_k g = s_k$.

Thus for all $k \in I$, $s_k rs = s_k$ and consequently since $(L, s_i)_I$ is a monsource, $rs=1_L$. Thus L is in $\overline{B}$ since $\overline{B}$ is isomorphism-closed.

H. Herrlich and G. Strecker [4, Theorem 7] have shown that an epireflective subcategory of any category is strongly closed under limits. In the case of complete categories we obtain as a corollary to Theorem 2.3 a similar result.

Corollary 2.4: If $\overline{C}$ is a complete category and $\overline{B}$ is an epireflective subcategory of $\overline{C}$, then $\overline{B}$ is strongly closed under limits.

Proof: Theorem 1.18 shows that $\overline{B}$ is closed under limits. Proposition 1.19 shows that $\overline{B}$ is regular hereditary in $\overline{C}$. Now apply Theorem 2.3.

We also obtain as a corollary to Theorem 2.3 the following categorical result which generalizes a topological result by van der Slot [7].

Theorem 2.5: If $\overline{C}$ is a complete category and $\overline{B}$ is a subcategory of $\overline{C}$ which is closed under products and is regular hereditary in $\overline{C}$, then $\overline{B}$ is strongly closed under limits.

Proof: It is well known that if a subcategory is closed under products and equalizers, then it is closed under limits. Now apply Theorem 2.3 to conclude the proof.

Corollary 2.6: [van der Slot: 7, Lemma 1.1.2] Let P be a property of Hausdorff spaces which is closed hereditary and productive. If $\{X_\alpha | \alpha \in A\}$ is a collection of subsets of a Hausdorff space and if each X_α satisfies property P, then $D = \cap\{X_\alpha | \alpha \in A\}$ satisfies property P.

Proof: Let $\overline{C}$ be the category of Hausdorff spaces and $\overline{B}$ the subcategory of all Hausdorff spaces satisfying the property P. It is clear that $\overline{B}$ is closed under products. By Theorem 1.26, $\overline{B}$ is regular hereditary in $\overline{C}$. Apply Theorem 2.5 and the result follows.

There are many examples of categorical situations in which a category $\overline{C}$ contains a subcategory $\overline{B}$ which is regular hereditary in $\overline{C}$. If P is any hereditary topological property and $\overline{B}$ is the subcategory of the category Top, of topological spaces, containing all the spaces with property P, then $\overline{B}$ is regular hereditary in Top. If Haus is the category of Hausdorff spaces and P is a closed hereditary property, then all the Hausdorff spaces satisfying property P form a regular hereditary subcategory of Haus.

We have seen that the regular hereditary condition is sufficient to guarantee that closure under limits is equivalent to strong closure under limits. We now show that there are other conditions that will give this result.

Before proceeding to any further results, we show the independence of the notions presented in Definitions 2.1 and 2.2.

We now exhibit a category $\overline{C}$ and a subcategory $\overline{B}$ of $\overline{C}$ such that $\overline{C}$ is subspanned by $\overline{B}$ but $\overline{B}$ is not regular hereditary in $\overline{C}$. Let $\overline{C}$ be the category of completely regular Hausdorff spaces and $\overline{B}$ the category of connected completely regular Hausdorff spaces. It is well known that every completely regular space can be embedded in a product of intervals; thus $\overline{C}$ is subspanned by $\overline{B}$. By Theorem 1.26, we see that regular subobjects in $\overline{C}$ are closed subspaces, and it is well known that closed subspaces of connected spaces are not necessarily connected; thus $\overline{B}$ is not regular hereditary in $\overline{C}$.

To show a subcategory $\overline{B}$ of a category $\overline{C}$ which is regular hereditary in $\overline{C}$, but fails to have $\overline{C}$ subspanned by $\overline{B}$, we let $\overline{C}$ be the category of Hausdorff spaces and $\overline{B}$ the category of discrete spaces. Clearly by Theorem 1.26 $\overline{B}$ is regular hereditary in $\overline{C}$, but not every Hausdorff space is the domain of a continuous one to one function with range contained in a discrete space. Thus $\overline{C}$ is not subspanned by $\overline{B}$.

We now prove a sequence of lemmas which leads to the next major theorem concerning strong closure under limits.

Lemma 2.7: Let $\overline{C}$ be a complete category and $\overline{B}$ a subcategory of $\overline{C}$ such that $\overline{C}$ is subspanned by $\overline{B}$; then the following are equivalent:

(i) $\overline{B}$ is closed under intersections,

(ii) $\overline{B}$ is strongly closed under intersections.

Proof: It suffices to prove that (i) implies (ii). Let $\{m_i : B_i \longrightarrow Y\}_I$ be a set of monomorphisms in $\overline{C}$ with each B_i an object in $\overline{B}$. Let $(L, k_i, d)_I$ be the intersection in $\overline{C}$ of this set of monomorphisms. We show that the object L is in the subcategory $\overline{B}$.

Since $\overline{C}$ is subspanned by $\overline{B}$, there exists a monomorphism $m: Y \longrightarrow B$ in $\overline{C}$ with B an object in $\overline{B}$. The family $\{mm_i : B_i \longrightarrow B\}_I$ is a set of monomorphisms in $\overline{B}$. Let $(T, t_i, f)_I$ be the intersection in $\overline{C}$ of this family. By (i), the object T is in the subcategory $\overline{B}$. To show that L is an object in $\overline{B}$ it suffices to prove that L is isomorphic to T.

Now $m(m_i t_i) = (mm_i) t_i = (mm_j) t_j = m(m_j t_j)$ for all i and j in I. The morphism m is a monomorphism in $\overline{C}$; thus $m_i t_i = m_j t_j$ for all $i, j \in I$. Consequently, there exists a unique morphism $h: T \longrightarrow L$ such that $k_i h = t_i$ for all i in I. Moreover, we have $(mm_i) k_i = m(m_i k_i) = m(m_j k_j) = (mm_j) k_j$ for any i and j in I. Thus there exists a unique morphism $k: L \longrightarrow T$ such that $t_i k = k_i$ for each $i \in I$ and the following diagram

$$\begin{tikzcd}
 & B_i \arrow[ddr, "m_i"] & & \\
L \arrow[ur, "k_i"] \arrow[dr, "k"] & & & \\
 & T \arrow[uu, "t_i"'] \arrow[dl, "h"'] \arrow[dd, "t_i"] & Y \arrow[r, "m"] & B \\
L \arrow[dr, "k_i"'] & & & \\
 & B_i \arrow[uur, "m_i"'] & &
\end{tikzcd}$$

commutes.

All that remains is to show that $kh=1_T$ and that $hk=1_L$. Now $t_i(kh)=(t_ik)h=k_ih=t_i=t_i1_T$ and since $(T,t_i)_I$ is a monosource in $\overline{C}$, we have that $kh=1_T$. Similarly, $k_i(hk)=(k_ih)k=t_ik=k_i=k_i1_L$ and $(L,k_i)_I$ is a monosource in $\overline{C}$. Thus $hk=1_L$. Since we only consider isomorphism-closed categories the lemma follows.

The proof of the following lemma is an adaptation of part of the proof of Theorem 8 in [4].

Lemma 2.8: If $\overline{C}$ is a finitely complete category and $\overline{B}$ is a subcategory of $\overline{C}$ which is strongly closed under the formation of finite intersections, then $\overline{B}$ is regular hereditary in $\overline{C}$.

Proof: Let $f,g:B\longrightarrow X$ be morphisms in $\overline{C}$, with B an object in $\overline{B}$. We wish to show that the equalizer of f and g is an object of $\overline{B}$. Let (P,p_B,p_X) be the product in $\overline{C}$ of the objects X and B with projection morphisms p_B and p_X. Let $\overline{f}$ be the unique morphism from B to P such that the following diagram

$$\begin{tikzcd}
X & B \arrow[l, "f"'] \arrow[r, "1_B"] \arrow[d, "\overline{f}"] & B \\
& P \arrow[ul, "p_X"] \arrow[ur, "p_B"'] &
\end{tikzcd}$$

commutes. Define $\overline{g}:B\longrightarrow P$ in an analogous fashion.

Let h and k be morphisms such that $\overline{f}h=\overline{f}k$. Then $1_Bh=(p_B\overline{f})h=p_B(\overline{f}h)=p_B(\overline{f}k)=(p_B\overline{f})k=1_Bk$; hence $\overline{f}$ is a monomorphism in $\overline{C}$. Likewise $\overline{g}$ is a monomorphism in $\overline{C}$. Let

(Q,r,s) be the intersection of $(B,\overline{f})$ and $(B,\overline{g})$ in $\overline{C}$; then Q is an object in $\overline{B}$ by the assumption that $\overline{B}$ is strongly closed under the formation of finite intersections. Thus the following diagram

$$\begin{tikzcd}
Q \arrow[r, "r"] \arrow[d, "s"'] & B \arrow[d, "\overline{f}"] \\
B \arrow[r, "\overline{g}"'] & P
\end{tikzcd}$$

commutes.

Now $\overline{f}r=\overline{g}s$; so $r=1_B r=(p_B\overline{f})r=p_B(\overline{f}r)=p_B(\overline{g}s)=(p_B\overline{g})s=1_B s=s$. Hence $gr=(p_X\overline{g})r=p_X(\overline{g}r)=p_X(\overline{f}r)=(p_X\overline{f})r=fr$. Moreover, if $u:E\longrightarrow B$ is a morphism such that $fu=gu$, then $p_X(\overline{f}u)=(p_X\overline{f})u=fu=gu=(p_X\overline{g})u=p_X(\overline{g}u)$ and $p_B(\overline{f}u)=(p_B\overline{f})u=1_B u=(p_B\overline{g})u=p_B(\overline{g}u)$. Since (P,p_B,p_X) is a monosource in $\overline{C}$, it follows that $\overline{f}u=\overline{g}u$. Thus there exists a unique morphism $t:E\longrightarrow Q$ such that the following diagram

$$\begin{tikzcd}
E \arrow[dr, "t"] \arrow[drr, "u"] \arrow[ddr, "u"'] & & \\
& Q \arrow[r, "r"] \arrow[d, "r"] & B \arrow[d, "\overline{f}"] \\
& B \arrow[r, "\overline{g}"'] & P
\end{tikzcd}$$

commutes. Thus (Q,r) is the equalizer of f and g and the proof is complete.

We are now prepared to state the second major result showing conditions for closure under limits to be equivalent to strong closure under limits.

Theorem 2.9: Let $\overline{C}$ be a complete category and $\overline{B}$ a subcategory of $\overline{C}$, such that $\overline{C}$ is subspanned by $\overline{B}$; then the following are equivalent:

(i) $\overline{B}$ is closed under limits,
(ii) $\overline{B}$ is strongly closed under limits.

Proof: It suffices to prove that (i) implies (ii). If (i) holds then $\overline{B}$ is closed under the formation of intersections, and by Lemma 2.7, $\overline{B}$ is strongly closed under the formation of intersections. By Lemma 2.8, $\overline{B}$ is also regular hereditary in $\overline{C}$. Now apply Theorem 2.3 and the theorem follows.

H. Herrlich and G. Strecker [4, Theorem 8] have shown that epireflective subcategories and subcategories which are strongly closed under limits are equivalent in certain categories. We obtain similar results in this direction, in particular by showing special cases where reflections are epireflections.

Theorem 2.10: Let $\overline{C}$ be a complete, well-powered and co-well-powered category which has the property that every extremal monomorphism is a regular monomorphism. Then if $\overline{B}$ is a subcategory of $\overline{C}$ which is regular hereditary in $\overline{C}$ the following are equivalent:

(i) $\overline{B}$ is epireflective in $\overline{C}$,
(ii) $\overline{B}$ is reflective in $\overline{C}$,
(iii) $\overline{B}$ is closed under limits,
(iv) $\overline{B}$ is strongly closed under limits.

Proof: That (i) implies (ii) is clear. P. Freyd [2] has shown that (ii) implies (iii), (1.18). Suppose that (iii) holds. Since $\overline{B}$ is regular hereditary in $\overline{C}$, Theorem 2.3 implies that $\overline{B}$ is strongly closed under limits in $\overline{C}$. If (iv) holds, then $\overline{B}$ is closed under products and regular subobjects and since every extremal monomorphism is a regular monomorphism, we conclude that $\overline{B}$ is extremal hereditary in $\overline{C}$. Now apply Theorem 1.20 to obtain that $\overline{B}$ is epireflective in $\overline{C}$. The theorem now follows.

Corollary 2.11: Let $\overline{C}$ be a complete, well-powered and co-well-powered category which has the property that every extremal monomorphism is a regular monomorphism. Then if $\overline{C}$ is subspanned by $\overline{B}$ the following are equivalent:

(i) $\overline{B}$ is epireflective in $\overline{C}$,
(ii) $\overline{B}$ is reflective in $\overline{C}$,
(iii) $\overline{B}$ is closed under limits,
(iv) $\overline{B}$ is strongly closed under limits.

Proof: Theorem 1.18 shows that if (ii) holds, then $\overline{B}$ is closed under limits. Thus if any of (i), (ii) or (iv) holds it is clear that (iii) holds. By Lemmas 2.7 and 2.8, if (iii) holds and $\overline{C}$ is subspanned by $\overline{B}$, then $\overline{B}$ is regular hereditary in $\overline{C}$. Consequently if any of the above four conditions hold, then $\overline{B}$ is regular hereditary in $\overline{C}$ and we may apply Theorem 2.10 to conclude the proof.

H. Herrlich and G. Strecker [6] have given a "topological proof" that the only reflective subcategory of the category *Haus*, of Hausdorff spaces, which contains the minimal Hausdorff spaces or the H-closed spaces is *Haus* itself. We present here, using the notions presented above, a simple categorical proof of this result.

Theorem 2.12: Let $\overline{C}$ be a category and E a class of epimorphisms in $\overline{C}$. Suppose that every morphism f in $\overline{C}$ has a factorization $f=me$ where $e\epsilon E$ and m is a monomorphism. Let $\overline{B}$ be a subcategory (not necessarily full) of $\overline{C}$ such that:

(i′) $\overline{C}$ is regularly subspanned by $\overline{B}$,

(ii′) whenever $e:C\longrightarrow B$ is a regular monomorphism in $\overline{C}$ with codomain in $\overline{B}$, there exist extremal monomorphisms with codomain in $\overline{B}$ which are equalized by e.

If $\overline{D}$ is a subcategory (not necessarily full) of $\overline{C}$ containing $\overline{B}$ such that:

(i) if X and Y are in $\overline{D}$ and $f:X\longrightarrow Y$ is an extremal monomorphism, then f is in $\overline{D}$,

(ii) if f is in $\overline{D}$ and $f=me$ with $e\epsilon E$, then e is in $\overline{D}$,

then the only reflective subcategory of $\overline{D}$ containing $\overline{B}$ is $\overline{D}$ itself.

Proof: Clearly $\overline{D}$ is a reflective subcategory of itself containing $\overline{B}$.

Let $\overline{A}$ be a reflective subcategory of $\overline{D}$ which contains $\overline{B}$. For each object X of $\overline{D}$ let $r:X \longrightarrow X_{\overline{A}}$ be the reflection morphism for X. By (i′), there exists a regular monomorphism $e:X \longrightarrow B$ in $\overline{C}$ with codomain B in $\overline{B}$. Let $h:X_{\overline{A}} \longrightarrow B$ be the unique morphism in $\overline{D}$ such that the following diagram

$$\begin{tikzcd}
X \arrow[r, "r"] \arrow[dr, "e"'] & X_{\overline{A}} \arrow[d, "h"] \\
 & B
\end{tikzcd}$$

commutes. By (ii′), there exist extremal monomorphisms $n,m:B \longrightarrow B'$ in $\overline{C}$ with codomain in $\overline{B}$ which are equalized by e. By (i), n and m are in the category $\overline{D}$. Since B' is an object in $\overline{A}$ there exists a unique morphism $t:X_{\overline{A}} \longrightarrow B'$ such that the following diagram

$$\begin{tikzcd}
X \arrow[r, "r"] \arrow[dr, "ne=me"'] & X_{\overline{A}} \arrow[d, "t"] \\
 & B'
\end{tikzcd}$$

commutes.

Now $(nh)r=n(hr)=ne=me=m(hr)=(mh)r$, and from the definition of uniqueness in reflections we have $nh=mh$. Let $h=k\overline{e}$ be a factorization of h with $e \in E$ and k a monomorphism. By (ii), $\overline{e}$ is a morphism in D. Now $(nk)\overline{e}=n(k\overline{e})=nh=mh=m(k\overline{e})=(mk)\overline{e}$ and since $\overline{e}$ is an epimorphism, $nk=mk$. Thus there exists a unique morphism $p:T \longrightarrow X$ from the domain T of k to X, such that $ep=k$.

We show that p is an extremal monomorphism in $\overline{C}$ and thus by (i) a morphism in $\overline{D}$. Now $k(\bar{e}rp)=(k\bar{e})rp=h(rp)=(hr)p=ep=k=k1_T$ and since k is a monomorphism, $\bar{e}rp=1_T$. Thus p is a section in $\overline{C}$ and by the remark following Proposition 1.8, a regular monomorphism in $\overline{C}$. By (i), p is a morphism in $\overline{D}$.

We now will show that X is isomorphic to $X_{\overline{A}}$. First, $e(p\bar{e}r)=k(\bar{e}r)=(k\bar{e})r=hr=e=e1_X$. Thus $p\bar{e}r=1_X$ since e is a monomorphism. Moreover, $p\bar{e}$ is a morphism in $\overline{D}$ and therefore $rp\bar{e}$ is a morphism in $\overline{D}$ such that the following diagram

$$\begin{tikzcd}
X \arrow[r, "r"] \arrow[dr, "r"'] & X_{\overline{A}} \arrow[d, "rp\bar{e}"] \\
 & X_{\overline{A}}
\end{tikzcd}$$

commutes. Hence $rp\bar{e}=1_{X_{\overline{A}}}$. Thus X is isomorphic to $X_{\overline{A}}$ and $\overline{A}$ is equal to $\overline{D}$.

As an immediate corollary to this theorem we have the following result.

Theorem 2.13: Let $\overline{C}$ be a category, and E a class of epimorphisms in $\overline{C}$. Suppose that every morphism f in $\overline{C}$ has a factorization $f=me$ where $e\epsilon E$ and m is a monomorphism. Then if $\overline{B}$ is a subcategory of $\overline{C}$ such that:

(i) $\overline{C}$ is regularly subspanned by $\overline{B}$, and

(ii) whenever $e:C\longrightarrow B$ is a regular monomorphism in $\overline{C}$, there exist extremal monomorphisms which have codomain in $\overline{B}$ and are equalized by e.

Then the only reflective subcategory of $\overline{C}$ containing $\overline{B}$ is $\overline{C}$ itself.

Proof: Let $\overline{D}=\overline{C}$ in Theorem 2.12; the result follows immediately.

Before stating any topological corollaries to the previous results, we state some well known definitions.

Definition 2.14: A space S is called a minimal Hausdorff space provided that it is Hausdorff and there exists no strictly coarser Hausdorff topology on the underlying set S.

Definition 2.15: A space S is called H-closed provided that it is Hausdorff and that for every Hausdorff space Y and each continuous function $f:S\longrightarrow Y$, $f[S]$ is a closed subspace of Y.

We are now prepared to state the topological corollaries to our previous results.

Corollary 2.16: [Herrlich, Strecker: 6, Theorem 1] Let $\overline{D}$ be a subcategory of the category Haus such that:

(i) each minimal Hausdorff space is an object in $\overline{D}$,

(ii) if X and Y are objects of $\overline{D}$ and $f:X\longrightarrow Y$ is a closed embedding, then f is a morphism in $\overline{D}$,

(iii) if $f:X\longrightarrow Y$ is a morphism in $\overline{D}$, then $Z=f[X]$ is an object of $\overline{D}$ and $\overline{f}:X\longrightarrow Z$ defined by $\overline{f}(x)=f(x)$ is a morphism in $\overline{D}$.

Then the only reflective subcategory of $\overline{D}$ containing the minimal Hausdorff spaces is $\overline{D}$ itself.

Proof: Let $\overline{C}$ be the category Haus and $\overline{B}$ a subcategory of $\overline{C}$ containing all the minimal Hausdorff spaces. G. Strecker and E. Wattel [8] have shown that every Hausdorff space is a closed subspace of a minimal Hausdorff space; thus $\overline{C}$ is regularly subspanned by $\overline{B}$. It is clear that if E is the class of all onto maps, then the factorization condition of Theorem 2.12 is satisfied. Conditions (ii) and (iii) above obviously correspond to conditions (i) and (ii) of Theorem 2.12. All we need show is that condition (ii′) of Theorem 2.12 is satisfied to conclude the proof.

Let $e:C \longrightarrow B$ be an extremal monomorphism in $\overline{C}$ with the object B a minimal Hausdorff space. By Theorem 1.26, $e[C]$ is a closed subspace of B. Let U be the disjoint union of B with itself and $j_1:B \longrightarrow U$ and $j_2:B \longrightarrow U$ the injection maps. For each $b \in e[C]$ identify the two points $j_1(b)$ and $j_2(b)$. Let Z be the corresponding quotient space and f the quotient map from U to Z. It is clear that fj_1 and fj_2 are closed embeddings from B to the minimal Hausdorff space Z; thus fj_1 and fj_2 are extremal monomorphisms in $\overline{C}$. The proof of Theorem 1.26 shows that e is the equalizer of fj_1 and fj_2; thus condition (ii′) of Theorem 2.12 is satisfied. The theorem follows.

Every minimal Hausdorff space is an H-closed space; hence the following corollaries are immediate.

Corollary 2.17: [Herrlich, Strecker: 6, Corollary 1] Haus is the only reflective subcategory of itself containing the minimal Hausdorff spaces.

Corollary 2.18: [Herrlich, Strecker; 6, Corollary 1] $\underline{\text{Haus}}$ is the only reflective subcategory of itself containing the H-closed spaces.

To conclude this chapter we present a final result which generalizes a result by van der Slot [7].

Theorem 2.19: Let $\overline{A} \hookrightarrow \overline{B} \hookrightarrow \overline{C}$ be full embeddings with $\overline{C}$ finitely complete, $\overline{B}$ regular hereditary, with the property that the product of an object in $\overline{A}$ with an object in $\overline{B}$ is an object in $\overline{B}$. If

$$\begin{tikzcd}
E \arrow[r] \arrow[d] & T \arrow[d] \\
Q \arrow[r] & C
\end{tikzcd}$$

is a pullback square in $\overline{C}$ with T and Q in $\overline{B}$, then E must be in $\overline{B}$.

Proof: Let $r:T \longrightarrow C$ and $s:Q \longrightarrow C$ be two morphisms with T in $\overline{A}$ and Q in the category $\overline{B}$. Let (P,p_Q,p_T) be the product in $\overline{C}$ of T and Q. P is an object in $\overline{B}$. Let $e:E \longrightarrow P$ be the equalizer in $\overline{C}$ of the morphisms $rp_T, sp_Q:P \longrightarrow C$; then E is an object in $\overline{B}$, since $\overline{B}$ is regular hereditary in $\overline{C}$.

We claim that the following diagram

$$\begin{tikzcd}
E \arrow[r, "p_T e"] \arrow[d, "p_Q e"'] & T \arrow[d, "r"] \\
Q \arrow[r, "s"'] & C
\end{tikzcd}$$

is a pullback square in $\overline{C}$. Suppose that the following diagram

$$\begin{tikzcd} S \arrow[r, "n"] \arrow[d, "m"'] & T \arrow[d, "r"] \\ Q \arrow[r, "s"'] & C \end{tikzcd}$$

commutes. Then by the definition of product there exists a unique morphism $t:S \longrightarrow P$ such that $p_T t=n$ and $p_Q t=m$. Now $(sp_Q)t=s(p_Q t)=sm=rn=r(p_T t)=(rp_T)t$, and from the definition of equalizer there exists a morphism $q:S \longrightarrow E$ such that $eq=t$. Moreover, $p_T(eq)=p_T t=n$ and $p_Q(eq)=p_Q t=m$. The theorem follows.

Corollary 2.20: [van der Slot: 7, Theorem 1.2.4] Let P be a property of Hausdorff spaces which is inherited by closed subsets and invariant for the taking of finite topological products. If f is a continuous map from a space X with property P onto a Hausdorff space Y, then the inverse image under f of each subset of Y with property P also satisfies P.

Proof: Let $\overline{C}$ be the category of all Hausdorff spaces and $\overline{B}$ the subcategory of all spaces satisfying property P. Then the hypotheses of Theorem 2.19 are satisfied by letting $\overline{A}=\overline{B}$. If $f:X \longrightarrow Y$ is a continuous map and $i:A \longrightarrow Y$ is an inclusion function then the inverse image of A under f is easily shown to be the object T in the pullback square

$$\begin{tikzcd}
T \arrow[r] \arrow[d] & A \arrow[d, "i"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Thus if X and A are in $\overline{B}$, by Theorem 2.19, T is an object in $\overline{B}$.

Corollary 2.21: Let $\overline{N}$ be the category of normed linear spaces and continuous linear transformations. Let $\overline{B}$ be the category of Banach spaces. If f is a linear transformation from a Banach space X to a normed linear space Y, then the inverse image under f of each subspace of Y which is a Banach space is also a Banach space.

Proof: It is well known that the finite product of normed linear spaces is a normed linear space and that the category $\overline{N}$ has equalizers; thus $\overline{N}$ is finitely complete. We show that $\overline{B}$ is regular hereditary in $\overline{N}$.

Let $f,g:B \longrightarrow N$ be two linear transformations with B a Banach space. If $Z=\{x\epsilon B \mid f(x)=g(x)\}$, then Z is a closed subspace of the Banach space B, since all normed linear spaces are Hausdorff spaces. Thus Z is a Banach space. It is well known that (Z,i) where i is the inclusion map is the equalizer in $\overline{N}$ of the morphisms $f,g:B \longrightarrow N$. Thus B is regular hereditary in $\overline{N}$. The finite product of Banach spaces is a Banach space; therefore we may apply Theorem 2.19 to complete the proof.

CHAPTER III

Fitting Subcategories

Throughout this chapter we will be primarily concerned with the study of monoreflective and epireflective subcategories of complete categories. We will see how the property of regular hereditariness leads to new results in this area. We begin by proving a lemma and theorem which will be the basis for a third definition used throughout the remainder of this paper.

Lemma 3.1: If $\overline{A}\longrightarrow\overline{B}\longrightarrow\overline{C}$ are full embeddings with $\overline{A}$ monoreflective in $\overline{C}$ and $\overline{C}$ finitely complete, and if the intersection of a subobject (A,a) and a subobject (B,b) of an object T in $\overline{B}$ is always an object in $\overline{B}$ provided $A\epsilon\overline{A}$ and $B\epsilon\overline{B}$, then $\overline{B}$ is regular hereditary in $\overline{C}$.

Proof: Let $m:X\longrightarrow B$ be a regular monomorphism in $\overline{C}$ with B an object in $\overline{B}$. Let $r,s:B\longrightarrow C$ be two morphisms in $\overline{C}$ which are equalized by the morphism $m:X\longrightarrow B$. We wish to show that X is an object in $\overline{B}$ since B is an object in $\overline{B}$. Let $t_1:X\longrightarrow X_{\overline{A}}$, $t_2:B\longrightarrow B_{\overline{A}}$ and $t_3:C\longrightarrow C_{\overline{A}}$ be the reflection morphisms for the objects X, B and C respectively. Let $\overline{m},\overline{r}$ and $\overline{s}$ be the induced morphisms such that the following diagrams

$$\begin{tikzcd}
X \arrow[r, "m"] \arrow[d, "t_1"'] & B \arrow[r, "r"] \arrow[d, "t_2"'] & C \arrow[d, "t_3"] \\
X_{\bar{A}} \arrow[r, "\bar{m}"'] & B_{\bar{A}} \arrow[r, "\bar{r}"'] & C_{\bar{A}}
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, "m"] \arrow[d, "t_1"'] & B \arrow[r, "s"] \arrow[d, "t_2"'] & C \arrow[d, "t_3"] \\
X_{\bar{A}} \arrow[r, "\bar{m}"'] & B_{\bar{A}} \arrow[r, "\bar{s}"'] & C_{\bar{A}}
\end{tikzcd}$$

commute. Let (K,e) be the equalizer of the morphisms $\bar{r},\bar{s}:B_{\bar{A}} \longrightarrow C_{\bar{A}}$. Since $\bar{A}$ is monoreflective in $\bar{C}$, by Proposition 1.16, $\bar{A}$ is epireflective in $\bar{C}$. Thus by Proposition 1.19, K is an object in $\bar{A}$.

Now $(\overline{rm})t_1=\bar{r}(\bar{m}t_1)=\bar{r}(t_2m)=(\bar{r}t_2)m=(t_3r)m=t_3(rm)=t_3(sm)=(t_3s)m=(\bar{s}t_2)m=\bar{s}(\bar{m}t_1)=(\overline{sm})t_1$ and since t_1 is an epimorphism we have $\overline{sm}=\overline{rm}$. Thus there exists a unique morphism $a:X_{\bar{A}} \longrightarrow K$ such that $ea=\bar{m}$. Let

$$\begin{tikzcd}
L \arrow[r, "s_1"] \arrow[d, "s_2"'] & B \arrow[d, "t_2"] \\
K \arrow[r, "e"'] & B_{\bar{A}}
\end{tikzcd}$$

be the intersection in $\bar{C}$ of the subobjects (B,t_2) and (K,e) of $B_{\bar{A}}$. Since $B_{\bar{A}}$ is an object in $\bar{B}$, L is an object in $\bar{B}$. We show that L is isomorphic to X.

Since $t_2m=\bar{m}t_1$, there exists a morphism $f:X \longrightarrow L$ such that $s_1f=m$ and $s_2f=at_1$. Moreover, $t_3(rs_1)=(t_3r)s_1=(\bar{r}t_2)s_1=\bar{r}(t_2s_1)=\bar{r}(es_2)=(\bar{r}e)s_2=(\bar{s}e)s_2=\bar{s}(es_2)=\bar{s}(t_2s_1)=(\bar{s}t_2)s_1=(t_3s)s_1=$

$t_3(ss_1)$. Thus since t_3 is a monomorphism there exists a unique morphism $g\colon L \longrightarrow X$ such that $mg=s_1$. Now $e(at_1g)=(ea)t_1g=\overline{m}t_1g=t_2mg=t_2s_1=es_2$ and since e is a monomorphism, $at_1g=s_2$.

Now $s_1(fg)=(s_1f)g=mg=s_1$ and $s_2(fg)=(s_2f)g=at_1g=s_2$, so that the following diagram

$$\begin{tikzcd}
 & & B \\
L \arrow[rru, "s_1"] \arrow[r, "fg"'] \arrow[rrd, "s_2"'] & L \arrow[ru, "s_1"'] \arrow[rd, "s_2"] & \\
 & & K
\end{tikzcd}$$

commutes; thus by the definition of intersection we have $fg=1_L$. Also $m(gf)=(mg)f=s_1f=m1_X$ and since m is a monomorphism in $\overline{C}$ we have $gf=1_X$. Thus X is isomorphic to L, and since L is an object in $\overline{B}$, X must be in $\overline{B}$.

Theorem 3.2: Let $\overline{A} \longrightarrow \overline{B} \longrightarrow \overline{C}$ be full embeddings with $\overline{A}$ monoreflective in $\overline{C}$ and $\overline{C}$ finitely complete. Then the following are equivalent:

(i) if

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & A \arrow[d] \\
B \arrow[r] & C
\end{tikzcd}$$

is a pullback square in $\overline{C}$ and A is an object in $\overline{A}$ and B is an object in $\overline{B}$, then P must be an object in $\overline{B}$,

(ii) $\overline{B}$ is regular hereditary in $\overline{C}$ and the product of any object in $\overline{A}$ with any object in $\overline{B}$ must be an object in $\overline{B}$.

Proof: Suppose (i) holds. It is a well known categorical result that if (A,a) and (B,b) are subobjects of an object C in a category $\overline{C}$ then the intersection of these subobjects coincides with the following pullback square.

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & A \arrow[d, "a"] \\
B \arrow[r, "b"'] & C
\end{tikzcd}$$

Thus we apply Lemma 3.1 to conclude that $\overline{B}$ is regular hereditary in $\overline{C}$.

To complete the proof of (i) implies (ii), let A be an object in $\overline{A}$ and B an object in $\overline{B}$. We show that A×B is an object in $\overline{B}$. Since $\overline{C}$ is finitely complete, $\overline{C}$ has a terminal object T. Let $r:B \longrightarrow T$ and $s:A \longrightarrow T$ be the unique morphisms to T from B and A respectively. Then the following diagram

$$\begin{tikzcd}
A \times B \arrow[r, "p_B"] \arrow[d, "p_A"'] & B \arrow[d, "r"] \\
A \arrow[r, "s"'] & T
\end{tikzcd}$$

commutes.

Suppose $f:Q \longrightarrow B$ and $g:Q \longrightarrow A$ are morphisms such that the following diagram

$$\begin{tikzcd}
Q \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "r"] \\
A \arrow[r, "s"'] & T
\end{tikzcd}$$

commutes. From the definition of product there exists a unique morphism $t:Q \longrightarrow A \times B$ such that $p_B t = f$ and $p_A t = g$. Consequently

$$\begin{tikzcd}
A \times B \arrow[r, "p_B"] \arrow[d, "p_A"'] & B \arrow[d, "r"] \\
A \arrow[r, "s"'] & T
\end{tikzcd}$$

is a pullback square, and by (i) $A \times B$ is an object in the category $\overline{B}$.

To show that (ii) implies (i), suppose that A is an object in $\overline{A}$ and B is an object in $\overline{B}$. Let $f:A \longrightarrow C$ and $g:B \longrightarrow C$. $A \times B$ is an object in $\overline{B}$ by (ii), hence for the morphism pair $fp_A, gp_B : A \times B \longrightarrow C$, let (E,e) be their equalizer in $\overline{C}$. Then by (ii) E is an object in $\overline{B}$. The proof of Theorem 2.19 shows that

$$\begin{tikzcd}
E \arrow[r, "p_A e"] \arrow[d, "p_A e"'] & A \arrow[d, "f"] \\
B \arrow[r, "g"'] & C
\end{tikzcd}$$

is a pullback square. The proof is now complete.

We are now prepared to state our next definition.

Definition 3.3: If $\overline{A} \longrightarrow \overline{B} \longrightarrow \overline{C}$ are full embeddings and satisfy properties (i) and (ii) in Theorem 3.2, we say that $\overline{B}$ is $\overline{A}$-fitting in $\overline{C}$.

The following theorem relates fitting to closure under limits and strong closure under limits. We have seen earlier how strong closure under limits often gives rise to epireflections (Theorem 2.10). We will now establish a sequence of results which indicates that fitting conditions also lead to epireflections.

Theorem 3.4: If $\overline{A} \longrightarrow \overline{B} \longrightarrow \overline{C}$ are full embeddings with $\overline{A}$ monoreflective in $\overline{C}$ and $\overline{C}$ complete, then the following are equivalent:

(i) $\overline{B}$ is $\overline{A}$-fitting in $\overline{C}$ and closed under intersections,

(ii) $\overline{B}$ is closed under products and is regular hereditary in $\overline{C}$,

(iii) $\overline{B}$ is closed under products, equalizers and intersections of equalizers,

(iv) $\overline{B}$ is closed under limits,

(v) $\overline{B}$ is strongly closed under limits,

(vi) $\overline{B}$ is closed under intersections and finite products,

(vii) $\overline{B}$ is closed under intersections, and the product of any object in $\overline{A}$ with any object in $\overline{B}$ is an object in $\overline{B}$.

Proof: The proof will follow the sequence (i⇒ii⇒iii⇒iv⇒v⇒vi⇒vii⇒i). Suppose $\overline{B}$ is $\overline{A}$-fitting and closed under intersections. Let $(B_i)_I$ be a set-indexed family of objects in $\overline{B}$ and $(P,p_i)_I$ their product in $\overline{C}$. Let $m:P\longrightarrow P_{\overline{A}}$ be the reflection morphism for P, and for each i in I let $m_i:B_i\longrightarrow B_{i_{\overline{A}}}$ be the reflection morphism for B_i.

Now for each $i\epsilon I$ there exists a unique morphism $t_i:P_{\overline{A}}\longrightarrow B_{i_{\overline{A}}}$ such that the following diagram

$$\begin{tikzcd}
P \arrow[r, "m"] \arrow[d, "p_i"'] & P_{\overline{A}} \arrow[d, "t_i"] \\
B_i \arrow[r, "m_i"'] & B_{i_{\overline{A}}}
\end{tikzcd}$$

commutes for each $i\epsilon I$.

For each $i\epsilon I$, B_i is an object in $\overline{B}$ and $P_{\overline{A}}$ is an object in $\overline{A}$. Thus for each $i\epsilon I$ let (P_i,s_i,r_i) be the pullback of the morphisms $t_i:P_{\overline{A}}\longrightarrow B_{i_{\overline{A}}}$ and $m_i:B_i\longrightarrow B_{i_{\overline{A}}}$. Since $\overline{B}$ is $\overline{A}$-fitting, the object P_i is an object in $\overline{B}$ for each i belonging to I. Now $t_i m=m_i p_i$ for each $i\epsilon I$, hence there exists a morphism $f_i:P\longrightarrow P_i$ such that the following diagram

$$\begin{tikzcd}
P_i \arrow[rrd, "r_i"] \arrow[rdd, "s_i"'] & & \\
& P \arrow[lu, "f_i"] \arrow[r, "m"'] \arrow[d, "p_i"] & P_{\overline{A}} \arrow[d, dash, "t_i"] \\
& B_i \arrow[r, "m_i"'] & B_{i_{\overline{A}}}
\end{tikzcd}$$

commutes for each $i\epsilon I$.

It is a well known categorical result that monomorphisms are preserved under pullbacks; consequently each morphism $r_i:P_i\longrightarrow P_{\overline{A}}$ is a monomorphism in the category $\overline{C}$.

Let $(T,k_i d)_I$ be the intersection of the morphisms $r_i:P_i \longrightarrow P_{\overline{A}}$. By (i), the object T is in the subcategory $\overline{B}$.

Since $r_i f_i = m$ for all $i \in I$, there exists a unique morphism $g:P \longrightarrow T$ such that $k_i g = f_i$ for each $i \in I$. Moreover, $(T,s_i k_i)_I$ is a source with codomain $(B_i)_{i \in I}$. Thus from the definition of product there exists a unique morphism $h:T \longrightarrow P$ such that $p_i h = s_i k_i$ for each $i \in I$.

For each $i \in I$ we have $p_i(hg) = (p_i h)g = (s_i k_i)g = s_i(k_i g) = s_i f_i = p_i = p_i 1_P$. Since $(P,p_i)_I$ is a monosource in $\overline{C}$, $hg = 1_P$. Hence the morphism $g:P \longrightarrow T$ is a section, and by the remark following Proposition 1.8, it is a regular monomorphism in $\overline{C}$. By (i), P is an object in $\overline{B}$, since the codomain of the regular monomorphism g is an object in $\overline{B}$. The proof of (i) implies (ii) is complete.

Suppose that $\overline{B}$ is closed under products and is regular hereditary in $\overline{C}$. Then to show that this implies (iii) it suffices to prove that $\overline{B}$ is closed under the intersection of equalizers. Let $(B_i,e_i)_I$ be a family of equalizers in $\overline{C}$ with B_i in $\overline{B}$ for each $i \in I$. Suppose that the codomain of each morphism e_i is the object B in the category $\overline{B}$. For each $i \in I$ there exists a pair of morphisms $f_i g_i : B \longrightarrow X_i$ which are equalized by $e_i : B_i \longrightarrow B$.

Let $(P,p_i)_I$ be the product of the X_i in $\overline{C}$. Then there exist morphisms $f,g:B \longrightarrow P$ such that the following diagram

$$\begin{tikzcd}
X_i & & P \arrow[ll, "p_i"'] \\
B \arrow[u, "f_i"] \arrow[urr, "f"'] \arrow[d, "g_i"'] \arrow[drr, "g"] & & \\
X_i & & P \arrow[ll, "p_i"]
\end{tikzcd}$$

commutes for each $i \in I$.

Let (Q,q) be the equalizer in $\overline{C}$ of the morphism pair $f,g:B \longrightarrow P$. Since q is a regular monomorphism in $\overline{C}$ with codomain B, an object in $\overline{B}$, by (ii) Q is an object in $\overline{B}$.

For each $i \in I$ we have $f_i q=(p_i f)q=p_i(fq)=p_i(gq)=g_i q$. Thus for each $i \in I$ there exists a unique morphism $k_i:Q \longrightarrow B_i$ such that the following diagram

$$\begin{tikzcd}
 & B_i \arrow[dr, "e_i"] & \\
Q \arrow[ur, "k_i"] \arrow[rr, "q"'] & & B
\end{tikzcd}$$

commutes. We show that $(Q,k_i,q)_I$ is the intersection of the morphisms $e_i:B_i \longrightarrow B$.

Suppose that $\{s_i:T \longrightarrow B_i\}_I$ is a family of morphisms such that $e_i s_i=e_k s_k$ for all i and k in I. Let $d=e_i s_i$ for any $i \in I$. Then $p_i(fd)=(p_i f)d=f_i d=f_i(e_i s_i)=(f_i e_i)s_i=(g_i e_i)s_i= g_i(e_i s_i)=g_i d=(p_i g)d=p_i(gd)$ for all $i \in I$. Thus since $(P,p_i)_I$

is a monosource in $\overline{C}$, $fd=gd$. Hence there exists a unique morphism $m:T\longrightarrow Q$ such that $qm=d$. However, $e_ik_im=qm=d=e_is_i$, and we have $k_im=s_i$ for each $i\in I$. Thus $(Q,k_i,q)_I$ is an intersection with Q an object in $\overline{B}$.

To show that (iii) implies (iv), suppose $\overline{B}$ is closed under products, equalizers and intersections of equalizers. Let $D:I\longrightarrow\overline{B}$ be a functor from a small category I into $\overline{B}$. Let $(L,k_i)_I$ be the limit of the functor $D:I\longrightarrow\overline{B}$ in $\overline{C}$. We need to show that L is an object in $\overline{B}$.

Form the product $(P,p_i)_I$ of the objects $D(i)$ in $\overline{B}$. Then by (iii), P is an object in the category $\overline{B}$. For each morphism $m:i\longrightarrow j$, let (E_m,e_m) be the equalizer of the morphism pair $D(m)p_i,p_j:P\longrightarrow D(j)$. For each m the object E_m is in the subcategory $\overline{B}$. Let $(T,s_m,d)_M$ be the intersection of the equalizers $e_m:E_m\longrightarrow P$. By (iii), T is an object in $\overline{B}$. We claim that $(T,p_id)_I$ is the limit of the functor $D:I\longrightarrow\overline{B}$.

Now $D(m)p_id=D(m)p_ie_ms_m=p_je_ms_m=p_jd$. Thus $(T,p_id)_I$ is a natural source for the functor $D:I\longrightarrow\overline{B}$. Suppose that $(Q,q_i)_I$ is also a natural source for $D:I\longrightarrow\overline{B}$. From the definition of product we obtain a unique morphism $h:Q\longrightarrow P$ such that $p_ih=q_i$ for each $i\in I$. Consequently $(D(m)p_i)h=D(m)q_i=q_j=p_jh$ for each $m:i\longrightarrow j$, and there exists a unique morphism $f_m:Q\longrightarrow E_m$ such that $e_mf_m=h$. Since $e_mf_m=h=e_nf_n$ for any n and m in I, there exists a unique morphism $v:Q\longrightarrow T$ such that $dv=h$. $(T,p_id)_I$ is therefore the limit of $D:I\longrightarrow\overline{B}$. Thus T is isomorphic to L and L must be an object in $\overline{B}$.

To see that (iv) implies (v), observe that $\overline{C}$ is subspanned by $\overline{B}$ and apply Theorem 2.9; (v) implies (vi) is clear as is (vi) implies (vii).

All that remains is to show that (vii) implies (i). It suffices to show that $\overline{B}$ is regular hereditary in $\overline{C}$. Since $\overline{A}$ is a full subcategory of $\overline{B}$ we may apply Lemma 3.1 to conclude that $\overline{B}$ is regular hereditary in $\overline{C}$.

At this point we will use the results just obtained to generalize to a categorical setting some further results of van der Slot [7]. We begin with a lemma.

Lemma 3.5: Let $\overline{A} \longrightarrow \overline{C}$ and $\overline{B} \longrightarrow \overline{C}$ be full embeddings with $\overline{C}$ complete and the terminal object of $\overline{C}$ an object in $\overline{B}$. If the product of any object in $\overline{A}$ with any object in $\overline{B}$ is an object in $\overline{B}$, then $\overline{A}$ is a full subcategory of $\overline{B}$.

Proof: Let A be any object of $\overline{A}$ and $(A\times T, p_T, p_A)$ the product of A with the terminal object of $\overline{C}$. $A\times T$ is an object in $\overline{B}$ since T is in $\overline{B}$. There exists a unique morphism $t: A \longrightarrow T$. Let u be the unique morphism from A to $A\times T$ such that the following diagram

$$\begin{tikzcd}
A & A\times T \arrow[l, "p_A"'] \arrow[r, "p_T"] & T \\
& A \arrow[ul, "1_A"] \arrow[u, "u"] \arrow[ur, "t"'] &
\end{tikzcd}$$

commutes.

Now $(p_T u)p_A = tp_A$ is a morphism from $A\times T$ to T, and since T is the terminal object of $\overline{C}$ it must be the case that $p_T u p_A = p_T$. Moreover, $p_T(up_A) = 1_A p_A = p_A$. Thus $up_A = 1_{A\times T}$, since $(A\times T, p_A, p_T)$ is a monosource in $\overline{C}$. Also $p_A u = 1_A$; thus A is isomorphic to $A \times T$ and A must therefore be an object in the category $\overline{B}$. That $\overline{A} \longrightarrow \overline{B}$ is a full embedding is clear.

Theorem 3.6: If $\overline{B} \longrightarrow \overline{C}$ and $\overline{A} \longrightarrow \overline{C}$ are full embeddings with $\overline{C}$ complete and $\overline{A}$ monoreflective in $\overline{C}$, then the following are equivalent:

(i) $\overline{B}$ contains the terminal object of $\overline{C}$, $\overline{B}$ is closed under intersections and the product of any object in $\overline{B}$ with any object in $\overline{A}$ is in $\overline{B}$,

(ii) $\overline{B}$ is closed under products, is regular hereditary in $\overline{C}$ and $\overline{A}$ is a full subcategory of $\overline{B}$,

(iii) $\overline{A} \longrightarrow \overline{B} \longrightarrow \overline{C}$ are full embeddings with $\overline{B}$ being $\overline{A}$-fitting and closed under intersections.

Proof: Suppose (i) holds; then by Lemma 3.5 $\overline{A}$ is a full subcategory of $\overline{B}$. Apply Theorem 3.4 (ii ⇔ vii) to conclude that (ii) holds.

Theorem 3.4, (ii ⇔ i), shows that (ii) implies (iii). To conclude the proof all we need show is that (iii) implies that $\overline{B}$ contains the terminal object of $\overline{C}$. This follows at once from $\overline{A}$ being monoreflective in $\overline{C}$ and consequently closed under limits.

We now state as a corollary to Theorem 3.6 the following result of van der Slot [7].

Theorem 3.7: [van der Slot: 7, Theorem 1.2.7] If P is a property of completely regular Hausdorff spaces, then the following are equivalent:

(i) every singleton space possesses property P, P is invariant under taking intersections, and property P satisfies the condition that the product of every space with property P and a compact Hausdorff space has property P,

(ii) P is a closed hereditary, productive property and is possessed by all compact Hausdorff spaces.

Proof: Let $\overline{C}$ be the category of all completely regular Hausdorff spaces, $\overline{B}$ the subcategory of $\overline{C}$ containing all spaces with property P, and $\overline{A}$ the category of all compact Hausdorff spaces. The theorem follows from the equivalence of (i) and (ii) in Theorem 3.6.

We now turn our attention to looking more closely at reflections with the objective of generalizing to a categorical setting a recent result by S. Franklin [1]. The major theorem we will present was suggested by G. Strecker and H. Herrlich. We begin by presenting a preliminary result.

Throughout the remainder of this paper, M will be a class of monomorphisms satisfying the following conditions:

$(\overline{\text{i}})$ M is closed under pullbacks and intersections,

$(\overline{\text{ii}})$ whenever fg is in M, then g belongs to M,

$(\overline{\text{iii}})$ whenever fg is an epimorphism and f belongs to M, then g is an epimorphism.

Theorem 3.8: Let $\overline{A} \longrightarrow \overline{B} \longrightarrow \overline{C}$ be full embeddings with $\overline{A}$ M-reflective in $\overline{C}$, $\overline{C}$ complete, well-powered, co-well-powered and having the property that every extremal monomorphism is a regular monomorphism. Then the following are equivalent:

(i) $\overline{B}$ is reflective in $\overline{C}$,

(ii) $\overline{B}$ is M-reflective in $\overline{C}$,

(iii) $\overline{B}$ is monoreflective in $\overline{C}$,

(iv) $\overline{B}$ is epireflective in $\overline{C}$,

(v) $\overline{B}$ is closed under limits,

(vi) $\overline{B}$ is closed under finite products and intersections,

(vii) $\overline{B}$ is $\overline{A}$-fitting and closed under intersections,

(viii) $\overline{B}$ is closed under products and is extremal hereditary in $\overline{C}$.

Proof: Suppose that (i) holds. Let C be an object of $\overline{C}$ and $r:C \longrightarrow C_{\overline{B}}$ the reflection morphism for C in $\overline{B}$. Since $\overline{A}$ is M-reflective in $\overline{C}$, there exists a reflective morphism $m:C \longrightarrow C_{\overline{A}}$ for C in $\overline{A}$ such that m is in M. $C_{\overline{A}}$ is an object in $\overline{B}$. Thus there exists a morphism $h:C_{\overline{B}} \longrightarrow C_{\overline{A}}$ such that the following diagram

$$\begin{tikzcd}
C \arrow[r, "r"] \arrow[dr, "m"'] & C_{\overline{B}} \arrow[d, "h"] \\
 & C_{\overline{A}}
\end{tikzcd}$$

commutes.

Thus m=hr, and by condition $(\overline{ii})$ on the class M, the morphism r must be in M.

Since M is a class of monomorphisms, (ii) implies (iii) is clear. Proposition 1.16 shows that (iii) implies (iv). (iv) implies (v) follows from Theorem 1.18. Theorem 3.4 shows that (v) through (viii) are equivalent and Theorem 1.20 shows that (viii) implies (i). This completes the proof.

Theorem 3.9: Let $\overline{A} \longrightarrow \overline{B} \longrightarrow \overline{C}$ be full embeddings with $\overline{A}$ M-reflective in $\overline{C}$, $\overline{C}$ complete, well-powered, and co-well-powered. If $\overline{B}$ is $\overline{A}$-fitting and $\overline{B}'$ is the epireflective hull of $\overline{B}$ in $\overline{C}$, then a $\overline{B}'$ reflection of a $\overline{C}$-object C can be obtained as the intersection of all the M-subobjects of the $\overline{A}$ reflection of C which belong to $\overline{B}$ and contain C.

Proof: Lemma 1.14 and Proposition 1.22 show that $\overline{B}'$ exists and that the objects in $\overline{B}'$ are extremal subobjects of products of objects in $\overline{B}$.

Let C be an object in $\overline{C}$ and $m: C \longrightarrow C_{\overline{A}}$ the M-reflection of C in $\overline{A}$. Let $F=((B_j, m_j))_J$ be the class of all M-subobjects of $C_{\overline{A}}$ which belong to $\overline{B}$ and contain C. To show that F is nonempty it suffices to observe that $1_{C_{\overline{A}}}$ is a morphism in M. This is the case since if $i: C_{\overline{A}} \longrightarrow (C_{\overline{A}})_{\overline{A}}$ is the M-reflection of $C_{\overline{A}}$ in $\overline{A}$ then $i = i1_{C_{\overline{A}}}$, and by condition (ii) on M, $1_{C_{\overline{A}}}$ is in M.

For each $j \in J$, there exists a morphism $q_j: C \longrightarrow B_j$ such that the following diagram

$$\begin{tikzcd}
C \arrow[rr, "m"] \arrow[dr, "q_j"'] & & C_{\overline{A}} \\
& B_j \arrow[ur, "m_j"'] &
\end{tikzcd}$$

commutes. $\overline{C}$ is well-powered. Therefore, let $\{C \xrightarrow{q_i} B_i \xrightarrow{m_i} C_{\overline{A}}\}_I$ be a representative set of factorizations of m where each (B_i, m_i) belongs to F. Let $(Q, k_i, d)_I$ be the intersection in $\overline{C}$ of the M-subobjects $(B_i, m_i)_I$ and let $n: C \longrightarrow Q$ be the unique morphism such that the following diagram

$$\begin{tikzcd}
C \arrow[rr, "m"] \arrow[dr, "q_i"] \arrow[ddr, "n"'] & & C_{\overline{A}} \\
& B_i \arrow[ur, "m_i"'] & \\
& Q \arrow[u, "k_i"] \arrow[uur, "d"'] &
\end{tikzcd}$$

commutes for each $i \in I$. Condition $(\overline{i})$ on the class M shows that the morphism d is in M, and by condition $(\overline{ii})$, the morphism n is in M. Corollary 2.4 and Theorem 3.4 show that Q is an object in $\overline{B}'$.

We claim that $n: C \longrightarrow Q$ is the M-reflection of C in $\overline{B}'$. The proof will be in two parts. Suppose that $t: C \longrightarrow B$ is any morphism in $\overline{C}$ with B an object in $\overline{B}$; we show that there exists a unique morphism $c: Q \longrightarrow B$ such that the following diagram

$$\begin{tikzcd}
C \arrow[r, "n"] \arrow[dr, "t"'] & Q \arrow[d, "c"] \\
& B
\end{tikzcd}$$

commutes.

Let $\overline{m}:B\longrightarrow B_{\overline{A}}$ be the M-reflection for B in $\overline{A}$, and $q:C_{\overline{A}}\longrightarrow B_{\overline{A}}$ the induced morphism such that the following diagram

$$\begin{tikzcd}
C \arrow[r, "m"] \arrow[d, "t"'] & C_{\overline{A}} \arrow[d, "q"] \\
B \arrow[r, "\overline{m}"'] & B_{\overline{A}}
\end{tikzcd}$$

commutes. Let

$$\begin{tikzcd}
P \arrow[r, "s"] \arrow[d, "r"'] & C_{\overline{A}} \arrow[d, "q"] \\
B \arrow[r, "\overline{m}"'] & B_{\overline{A}}
\end{tikzcd}$$

be the pullback of the morphisms $q:C\longrightarrow B_{\overline{A}}$ and $\overline{m}:B\longrightarrow B_{\overline{A}}$. Condition $(\overline{i})$ on M implies that the morphism $s:P\longrightarrow C_{\overline{A}}$ is in M. There exists a unique morphism $f:C\longrightarrow P$ such that sf=m and rf=t. Since $\overline{B}$ is $\overline{A}$-fitting, P is an object in $\overline{B}$. Since m is a monomorphism f is a monomorphism, thus sf=m is a factorization of m where (P,s) is an M-subobject of $C_{\overline{A}}$ containing C. Thus, up to isomorphism, $s=m_i$ and $f=q_i$ for some $i\epsilon I$. For clarity let $f=q_v$ and $s=m_v$ for some fixed $v\epsilon I$. Then the following diagram

$$\begin{tikzcd}
C \arrow[rr, "m"] \arrow[dr, "n"] \arrow[ddr, "q_v=f"'] \arrow[ddd, "t"'] & & C_{\overline{A}} \arrow[ddd, "q"] \\
 & Q \arrow[ur, "d"] \arrow[d, "k_v"] & \\
 & P \arrow[uur, "s=m_v"'] \arrow[dl, "r"] & \\
B \arrow[rr, "\overline{m}"'] & & B_{\overline{A}}
\end{tikzcd}$$

commutes. Now $\overline{m}(rk_v n)=(\overline{m}r)k_v n=(qs)k_v n=q(sk_v)n=qdn=qm=\overline{m}t$, which implies that $rk_v n=t$ since $\overline{m}$ is a monomorphism in $\overline{C}$. Condition $\overline{(iii)}$ on M shows that n is an epimorphism; thus rk_v is the unique morphism p such that $pn=t$.

Now suppose that $\overline{t}:C\longrightarrow B$ is any morphism in $\overline{C}$ with B an object in $\overline{B}'$. As we observed at the beginning of the proof, B is an extremal subobject of a product of objects in $\overline{B}$; therefore let $e:B\longrightarrow P$ be an extremal monomorphism in $\overline{C}$ where $(Z,p_o)_O$ is a product of objects $(B_o)_O$ in $\overline{B}$. For each $o\epsilon O$ there is a morphism $p_o e\overline{t}:C\longrightarrow B_o$, hence from the previous work we conclude that there exists a morphism $g_o:Q\longrightarrow B_o$ such that $g_o n=p_o e\overline{t}$. Consequently, from the definition of product there exists a unique morphism $u:Q\longrightarrow P$ such that $p_o u=g_o$ for each $o\epsilon O$. Now $(Z,p_o)_O$ is a monosource and since $p_o un=g_o n=p_o e\overline{t}$ we have $un=e\overline{t}$ and the following diagram

$$\begin{tikzcd}
& C \arrow[rrrr, "m"] \arrow[dl, "\overline{t}"'] \arrow[drr, "q_i"] \arrow[ddrr, "n"] & & & C_{\overline{A}} \\
B \arrow[d, "e"'] & & & B_i \arrow[ur, "m_i"] & \\
Z \arrow[dr, "p_o"'] & & & Q \arrow[lll, "u"'] \arrow[u, "k_i"] \arrow[uur, "d"'] \arrow[dll, "g_o"] & \\
& B_o & & &
\end{tikzcd}$$

commutes. Since

$$\begin{tikzcd}
C \arrow[r, "n"] \arrow[d, "\overline{t}"'] & Q \arrow[d, "u"] \\
B \arrow[r, "e"'] & P
\end{tikzcd}$$

is a commutative diagram with n an epimorphism and e an extremal monomorphism there exists (1.12) a morphism $b:Q \longrightarrow B$ such that $bn=\overline{t}$ and $eb=u$. Thus the following diagram

$$\begin{tikzcd}
C \arrow[r, "n"] \arrow[dr, "\overline{t}"'] & Q \arrow[d, "b"] \\
& B
\end{tikzcd}$$

commutes. That b is unique follows from n being an epimorphism. This completes the proof.

We now turn our attention to some topological applications of Theorem 3.9. If $\overline{C}$ is the category of completely regular Hausdorff spaces then $\overline{C}$ is complete, well-powered, and co-well-powered. The reader is referred to the work of H. Herrlich and G. Strecker [5] for a proof of the results. Moreover, if M is the class of all topological embedding maps, then M satisfies the conditions $(\overline{i})$, $(\overline{ii})$ and $(\overline{iii})$ we desire. Let $\overline{A}$ be the category of compact Hausdorff spaces, then $\overline{A}$ is M-reflective in $\overline{C}$, where the embedding of a completely regular space into the Stone-Čech compactification is the required M-reflection.

We now investigate what types of conditions will give rise to suitable categories which are $\overline{A}$-fitting. We turn our attention to topological properties which are inversely preserved under perfect maps. We prove some well known topological results before proceeding to state S. Franklin's result.

Definition 3.10: A map $p:X \longrightarrow Y$ between two topological spaces will be called perfect provided that it is closed, continuous and the inverse images of points are compact.

Proposition 3.11: If Y is a compact space and Z any Hausdorff space, then the projection map $p:Y\times Z \longrightarrow Z$ is a closed map.

Proof: Let F be a closed subset of the product $Y\times Z$; we show that $Z-p[F]$ is open in Z. Let $q_o \epsilon Z-p[F]$. Then $F \cap (Y\times\{q_o\}) = \emptyset$, hence for each point (y,q_o) in $Y\times\{q_o\}$ there exists an open neighborhood $O_y \times O^y$ containing (y,q_o) which does not intersect F. Since Y is compact, there is a finite subset S of Y such that $\{O_y | y \in S\}$ covers Y. Then $\cap\{O^y | y \in S\}$ is an open neighborhood of q_o which misses $p[F]$. Thus $Z-p[F]$ is open and p is a closed map.

Proposition 3.12: If B is a closed subspace of X, then the inclusion map is perfect.

Proof: The proof is clear.

Proposition 3.13: If A is a compact Hausdorff space and B is any Hausdorff space then $p:A\times B \longrightarrow B$ is a perfect map.

Proof: By Proposition 3.11, p is closed. That p is continuous is clear and that the inverse images of points are compact is also clear.

The previous propositions show that if P is any property of completely regular spaces which is inversely preserved under perfect maps and every singleton space has property P, then the category $\overline{B}$ of all completely regular spaces satisfying property P is $\overline{A}$-fitting, where $\overline{A}$ is the category of compact Hausdorff spaces.

If P is one of the following properties: compactness, countable compactness, local compactness, σ-compactness, or paracompactness, then P is inversely preserved under perfect maps. One could also allow $\overline{B}$ to be the category of Lindelöf spaces or topologically complete spaces. (See [1].)

Corollary 3.14: [Franklin, 1] If $\overline{B}$ is any category of completely regular spaces which contains the singleton spaces and is inversely preserved under perfect maps, then every object X in the epireflective hull of $\overline{B}$ can be obtained as the intersection of the class of subsets of the Stone-Čech compactification of X which contain X and are members of $\overline{B}$.

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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